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ABSTRACT

This is one of a series that is a collection of translations from the extensive Soviet literature of the past 25 years on research in the psychology of mathematics instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English. The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. The work of El'konin, Davydov, and Minskaya reported in this volume represents a start toward the alleviation of the lack of theory-based experimental investigations of mathematics learning and teaching. Chapter titles include: Learning Capacity and Age Level; Primary Schoolchildren's Intellectual Capabilities and the Content of Instruction; Logical and Psychological Problems of Elementary Mathematics as an Academic Subject; The Psychological Characteristics of the "Prenumerical" Period of Mathematics Instruction; and Developing the Concept of Number by Means of the Relationship of Quantities. (Author/MK)

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SOVIET STUDIES
IN THE
PSYCHOLOGY OF LEARNING
AND TEACHING MATHEMATICS

VOLUME VII

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EDUCATION & WELFARE
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EDUCATION

Mary L. Charles
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SCHOOL MATHEMATICS STUDY GROUP
STANFORD UNIVERSITY
AND
SURVEY OF RECENT EAST EUROPEAN
MATHEMATICAL LITERATURE
THE UNIVERSITY OF CHICAGO

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PSYCHOLOGY OF LEARNING
AND TEACHING MATHEMATICS

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VOLUME VII

CHILDREN'S CAPACITY FOR LEARNING MATHEMATICS

VOLUME EDITOR

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PREFACE

The series Soviet Studies in the Psychology of Learning and Teaching Mathematics is a collection of translations from the extensive Soviet literature of the past twenty-five years on research in the psychology of mathematical instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. The series is the result of a joint effort by the School Mathematics Study Group at Stanford University, the Department of Mathematics Education at the University of Georgia, and the Survey of Recent East European Mathematical Literature at the University of Chicago. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English.

Research achievements in psychology in the United States are outstanding indeed. Educational psychology, however, occupies only a small fraction of the field, and until recently little attention has been given to research in the psychology of learning and teaching particular school subjects.

The situation has been quite different in the Soviet Union. In view of the reigning social and political doctrines, several branches of psychology that are highly developed in the U.S. have scarcely been investigated in the Soviet Union. On the other hand, because of the Soviet emphasis on education and its function in the state, research in educational psychology has been given considerable moral and financial support. Consequently, it has attracted many creative and talented scholars whose contributions have been remarkable.*

Even prior to World War II, the Russians had made great strides in educational psychology. The creation in 1943 of the Academy of Pedagogical Sciences helped to intensify the research efforts and programs in this field. Since then the Academy has become the chief educational research and development center for the Soviet Union. One of the main aims of the Academy is to conduct research and to train research scholars in general and specialized education, in educational psychology, and in methods of teaching various school subjects.

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*A study indicates that 37.5% of all materials in Soviet psychology published in one year was devoted to education and child psychology. See Contemporary Soviet Psychology by Josef Brozek (Chapter 7 of Present-Day Russian Psychology, Pergamon Press, 1966).

The Academy of Pedagogical Sciences of the USSR comprises ten research institutes in Moscow and Leningrad. Many of the studies reported in this series were conducted at the Academy's Institute of General and Polytechnical Education, Institute of Psychology, and Institute of Defectology, the last of which is concerned with the special psychology and educational techniques for handicapped children.

The Academy of Pedagogical Sciences has 31 members and 64 associate members, chosen from among distinguished Soviet scholars, scientists, and educators. Its permanent staff includes more than 650 research associates, who receive advice and cooperation from an additional 1,000 scholars and teachers. The research institutes of the Academy have available 100 "base" or laboratory schools and many other schools in which experiments are conducted. Developments in foreign countries are closely followed by the Bureau for the Study of Foreign Educational Experience and Information.

The Academy has its own publishing house, which issues hundreds of books each year and publishes the collections *Izvestiya Akademii Pedagogicheskikh Nauk RSFSR* [Proceedings of the Academy of Pedagogical Sciences of the RSFSR], the monthly *Sovetskaya Pedagogika* [Soviet Pedagogy], and the bimonthly *Voprosy Psikhologii* [Questions of Psychology]. Since 1963, the Academy has been issuing collections entitled *Novye Issledovaniya v Pedagogicheskikh Naukakh* [New Research in the Pedagogical Sciences] in order to disseminate information on current research.

A major difference between the Soviet and American conception of educational research is that Russian psychologists often use qualitative rather than quantitative methods of research in instructional psychology in accordance with the prevailing European tradition. American readers may thus find that some of the earlier Russian papers do not comply exactly to U.S. standards of design, analysis, and reporting. By using qualitative methods and by working with small groups, however, the Soviets have been able to penetrate into the child's thoughts and to analyze his mental processes. To this end they have also designed classroom tasks and settings for research and have emphasized long-term, genetic studies of learning.

Russian psychologists have concerned themselves with the dynamics of mental activity and with the aim of arriving at the principles of the learning process itself. They have investigated such areas as: the development of mental operations; the nature and development of thought; the formation of mathematical concepts and the related questions of generalization, abstraction, and concretization; the mental operations of analysis and synthesis; the development of spatial perception; the relation between memory and thought; the development of logical reasoning; the nature of mathematical skills; and the structure and special features of mathematical abilities.

In new approaches to educational research, some Russian psychologists have developed cybernetic and statistical models and techniques, and have made use of algorithms, mathematical logic and information sciences. Much attention has also been given to programmed instruction and to an examination of its psychological problems and its application for greater individualization in learning.

The interrelationship between instruction and child development is a source of sharp disagreement between the Geneva School of psychologists, led by Piaget, and the Soviet psychologists.* The Swiss psychologists ascribe limited significance to the role of instruction in the development of a child. According to them, instruction is subordinate to the specific stages in the development of the child's thinking--stages manifested at certain age levels and relatively independent of the conditions of instruction.

As representatives of the materialistic-evolutionist theory of the mind, Soviet psychologists ascribe a leading role to instruction. They assert that instruction broadens the potential of development, may accelerate it, and may exercise influence not only upon the sequence of the stages of development of the child's thought but even upon the very character of the stages. The Russians study development in the changing conditions of instruction, and by varying these conditions, they demonstrate how the nature of the child's development changes in the process. As a result, they are also investigating tests of giftedness and are using elaborate dynamic, rather than static, indices.

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*See The Problem of Instruction and Development at the 18th International Congress of Psychology by N. A. Menchinskaya and G. G. Saburova, Sovetskaya Pedagogika, 1967, No. 1. (English translation in Soviet Education, July 1967, Vol. 9, No. 9.)

Psychological research has had a considerable effect on the recent Soviet literature on methods of teaching mathematics. Experiments have shown the student's mathematical potential to be greater than had been previously assumed. Consequently, Russian psychologists have advocated the necessity of various changes in the content and methods of mathematical instruction and have participated in designing the new Soviet mathematics curriculum which has been introduced during the 1967-68 academic year.

The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. This series should assist in opening up avenues of investigation to those who are interested in broadening the foundations of their profession, for it is generally recognized that experiment and research are indispensable for improving content and methods of school mathematics.

We hope that the volumes in this series will be used for study, discussion, and critical analysis in courses or seminars in teacher-training programs or in institutes for in-service teachers at various levels.

At present, materials have been prepared for fifteen volumes. Each book contains one or more articles under a general heading such as The Learning of Mathematical Concepts, The Structure of Mathematical Abilities and Problem Solving in Geometry. The introduction to each volume is intended to provide some background and guidance to its content.

Volumes I to VI were prepared jointly by the School Mathematics Study Group and the Survey of Recent East European Mathematical Literature, both conducted under grants from the National Science Foundation. When the activities of the School Mathematics Study Group ended in August, 1972, the Department of Mathematics Education at the University of Georgia undertook to assist in the editing of the remaining volumes. We express our appreciation to the Foundation and to the many people and organizations who contributed to the establishment and continuation of the series.

Jeremy Kilpatrick
Izaak Wirszup
Edward G. Begle
James W. Wilson

EDITORIAL NOTES

1. Bracketed numerals in the text refer to the numbered references at the end of each paper. Where there are two figures, e.g. [5:123], the second is a page reference. All references are to Russian editions, although titles have been translated and authors' names transliterated.

2. The transliteration scheme used is that of the Library of Congress, with diacritical marks omitted, except that Ю and Я are rendered as "yu" and "ya" instead of "iu" and "ia."

3. Numbered footnotes are those in the original paper, starred footnotes are used for editors' or translator's comments.

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INTRODUCTION

Leslie P. Steffe

Mathematics education lacks systematic experimental investigations of mathematics learning and teaching that are based on theory. Begle [1] has expressed the opinion that further substantive improvement in mathematics education will not take place until mathematics education is turned into an experimental science. Piaget [5] observed that no more was known in 1965 than was known in 1935 concerning what remains of the knowledge acquired in primary and secondary schools after time intervals, say, of five, ten, or twenty years. Piaget's observation is consistent with Begle's. Both are comments on the lack of theory-based experimental investigations of mathematics learning and teaching. The work of El'konin, Davydov, and Minskaya reported in this volume represents a start toward the alleviation of this condition. The experimental curriculum posited by these authors was generated by a thorough theoretical analysis and synthesis of the cognitive development of children, fundamental mathematical structures, and the content of mathematics instruction in the early school years. The empirical results obtained are particularly noteworthy in light of this analysis and synthesis and are potentially applicable not only to mathematics education, but to the psychology of childhood as well.

In the introductory chapter, El'konin and Davydov outline their position concerning the relation of instruction to the intellectual development of children, a position considerably elaborated in later chapters. Following Vygotskii, they view a child's mental development as being ultimately determined by the content of the knowledge studied. Researchers (notably Piaget) who study the development of mental operations generally concentrate on those mental operations which are maximally independent of specific subject matter. El'konin and Davydov criticize this approach because it leads to a view that the sources of mental development lie in the individual himself, independent of specific historical conditions of existence (including education), and because it leads to an absolute way of characterizing features of the

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child's mind according to age level. Consequently, they believe that existing intellectual capabilities can be studied only by making changes in the content of what children learn at school. In fact, the main task of their research was to study ways of designing academic subjects and children's school work so that much of it becomes "accessible" to the students. El'konin and Davydov do not link their work to Bruner's famous hypothesis that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development [2:33]." The particular weakness they see in Bruner's hypothesis is that it makes reference to abstract forms of teaching the fundamentals of any subject to a child of any age. It is not possible to characterize capabilities for learning with regard to age level in the abstract -- forms of instruction must be found that are suitable for each specific piece of content and given age level. They contend that Bruner was correct, however, in challenging the traditional absolute way of characterizing features of the child's mind according to age level.

One wonders, however, about the degree of divergence between the views of El'konin and Davydov on the one hand and the Genevans; on the other. First, Piaget [5:21] has clearly differentiated experimental pedagogy and psychology: experimental pedagogy is concerned less with the general and spontaneous characteristics of the child than with their modification through pedagogic processes. Second, in commenting on the value of development stages in educational science, Piaget [5:171] rejects the notion of inflexible stages characterized by invariable chronological age limits and permanent thought content. As an interactionist, Piaget [5:172-73] advocates that the mind's structural maturation and the child's individual experience each be considered as factors in intellectual development. Third, Piaget [4:16] has commented that mathematical structures can be learned if the structure you want to teach can be supported by simpler, more elementary structures. These three considerations are not completely inconsistent with El'konin and Davydov's basic position concerning development. Certainly, the Genevans' work has not been in experimental pedagogy but rather has dealt with the development of the child in the most general ways. But experimental pedagogy does not stand in opposition to cognitive development psychology. Rather, experimental pedagogy is complementary to it, with the potential of contributing knowledge to developmental processes. One of the most fecund areas for such potential contribution lies in the formation of mental operations (which may or may not have been studied by the Genevan school) -- that is, in understanding the contribution of instruction in school mathematics to the formation of mental operations.

In the second chapter, El'konin elaborates on the points made in the introductory chapter. Through analyses of the writings of Piaget, Blonskii, Zankov, and Vygotskii, El'konin formulates a basic hypothesis that a change in the content of instruction coupled with a corresponding change in the type of teaching will influence the chronological outline of the development of the child's intellect. The following are among the various basic ideas that lead to formulating this hypothesis.

A central issue is whether to characterize a given age level in terms of the processes for which development is concluded at that age level or in terms of the processes for which development is beginning at that age level. If the former point of view is adopted, then one is led to a conception of intellectual development as being inviolable and independent of the content and methods of presentation of subject matter. This point of view leads to exercises being presented to the children that demand only previously formed intellectual processes for solution. However, if the latter point of view is adopted (and it is by El'konin) then the content of instruction becomes exceedingly important. Following Vygotskii, El'konin believes that the development of the psychological processes for learning mathematics do not precede instruction in mathematics, but that the characteristics for learning new content are formed in the process of learning it. The emphasis, however, is not placed on the method of instruction (here is where the author diverges from Bruner's hypothesis) but on the content of instruction. The teaching methods are to be organically connected with the content and are to create a bond between the child and society, where the teacher represents the knowledge accumulated by society and is not merely the child's colleague.

In the third chapter, Davydov outlines the three basic structures of the Bourbaki -- algebraic structures, structures of order, and topological structures -- as a basis for structuring school mathematics from the beginning. Davydov is well aware of the role of these three basic structures in cognitive development theory espoused vis-a-vis Piaget. In fact, it is here that he genuinely uses the results of psychological research in structuring the experimental curricula. He does so, however, in full awareness of the difficulty of determining explicitly how mathematical structures and genetic structures of thought are related. In what way are mathematical structures a continuation of previously formed genetic structures? Nevertheless, curricula designed on the basis of initial mathematical structures are supported by Piaget's theory. If the assumption is made that the child's mathematical thought develops within the very process of the formation of concrete operations, then because concrete operations are to be considered as operative structures, curricula based on mathematical structures can be introduced at the beginning of the period of concrete operations. The possibility then exists that the onset of formal operations can be hastened through study of mathematical structure. Experimental pedagogy and psychological theory merged naturally in Davydov's analysis, but nevertheless remained distinct (or at least not completely married) by virtue of the problems being studied. This use of knowledge and theory gained through psychological research is a welcome relief from harsh rejections, such as Menchinskaya's [3:78].

At the beginning of the fourth chapter, Davydov discusses a myriad of issues concerned with the traditional mathematics course in Soviet schools, issues brought about in part through viewing mathematics from a structural standpoint. Through discussion of these issues, Davydov concludes that the concept of quantity needs analysis in the search for the "common root of the branching tree of mathematics." A quantity is defined as any set of elements for which criteria of comparison have been established satisfying eight postulates of comparison. Quantity, then, is a particular instance of the structure of order. Kolmogorov, however, restricts the notion of quantity so that the real numbers become quantities. Starting from this more restricted notion of quantity, Davydov gives a detailed description of the content of his experimental curriculum for four months of the first grade (seven year olds) and the organization of instruction in that curriculum. This curriculum pertaining to the concept of quantity is organized around eight topics, among which are the properties of equality and inequality (Topic III). The experimental methodology employed includes actually teaching the material in experimental classrooms. The results reported are largely anecdotal and are reported on the most characteristic features of the teaching process and its results, the features which are typical of the various classes. Objective data are given, however, on various problems given to children at the end of instruction. Generally, the data reported (anecdotal and objective) were very favorable.

In the final chapter, Minskaya describes the experimentation on the concept of number. Number is studied, using the previous material as a foundation, where major attention is given to studying number as a relationship of a given quantity to a unit of measurement. The results reported are also quite favorable, consisting of anecdotal data and objective data from the administration of various problems.

As highly provocative as the volume is, there are shortcomings. What evidence should one accept that children have learned operational structures? Piaget [4:17-18] has identified three criteria -- (a) Is the learning lasting? (b) How much generalization is possible? (c) In the case of each learning experience, what was the operational level of the subject before the experience and what more complex structures has this learning succeeded in achieving? A fourth piece of evidence, which seems necessary, concerns the organization of the "learned" structure in the child's mind.

No data are presented with regard to the third criterion. While the authors assumed concrete operations, the age level of the children would suggest variability of stage level. Moreover, the system of problems that were used in the experiments and the experimental methodology are highly disputable with regard to each of the above general criteria. In the face of such disputation, one can only conjecture as to the substantive contribution of the instruction to the children's mental development. Here, it must be noted that Talyzina [6:22] considers the instructional program not beyond the powers of children in the first grade and that it was mastered fully by the majority of pupils.

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LEARNING CAPACITY AND AGE LEVEL: INTRODUCTION*

D. B. El'konin and V. V. Davydov

There has been much discussion in recent years, both here and abroad, concerning methods of improving content of grade school education. In the discussion, attention has been centered on finding ways of bridging the gap between school curricula and modern scientific knowledge. *Elementary instruction*, with its resources for broadening and deepening education as a whole, is an important instrument to be used in solving the complex problems in bridging this gap. Up to now, however, the resources of elementary instruction have not been used to their full extent. Furthermore, attempts which *have* been made to substantially alter elementary instruction have met with a number of serious objections. One particular objection has to do with the traditional conception of *age level as a factor in the mental activity of primary school children*. Age level supposedly radically limits the range of information and concepts which grade school children can learn.

For several years, the research group at the laboratory for the psychology of the primary school child at the Institute of Psychology of the Academy of Pedagogical Sciences has been studying age level as a factor in the intellect of primary school children. Issues confronting this research group are: Does age level in fact drastically limit curriculum content and the ways it can be altered?; Are there capabilities for intellectual development at the primary grades which remain undetected? How are these assumed capabilities related to ways of designing academic subjects?

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*From *Learning Capacity and Age Level: Primary Grades*, edited by D. B. El'konin and V. V. Davydov, Moscow, Prosveshchenie, 1966, pp. 3-12. Translated by Anne Bigelow.

In this book, we outline our approach to these issues and set forth some specific results of our attempts to resolve them. Our research has been guided by the theoretical statements formulated as long ago as the 1930's by the Soviet psychologist L. S. Vygotskii which, in our view, reveal the basic long-term course of development of educational and child psychology. Vygotskii's statements are being corroborated and developed further in contemporary theoretical and experimental psychological research (A. N. Leont'ev [7]; P. Ya. Gal'perin, A. V. Zaporozhets, and D. B. El'konin [6]; and others). For us the key statement was that in the final analysis, a pupil's mental development is determined by the content of what he is learning. Existing intellectual capabilities must therefore be studied primarily by making certain changes in what children learn at school.

When investigating mental development, psychologists tend to study certain mental operations which are maximally independent of specific subject matter (this is essentially the approach taken by psychologists of the school of J. Piaget). Vygotskii was critical of this research method:

> The attempt to analyze the mental development of the child by making a careful division between what comes from development and what comes from learning, and then taking the results of both these processes in their pure and isolated form is typical of this approach. Since not a single researcher has yet been able to do this, imperfections in the methodological procedures being used are usually cited as the cause, and an attempt is made to compensate for their shortcomings by using abstraction to divide the child's intellectual characteristics into those arising from development and those resulting from learning [9:252].

We are in agreement with Vygotskii that attempts at such a "division" not only are impracticable, but hinder fruitful study of the actual conditions and principles governing the child's mental development. The very fact that a child has mastered certain material is the most important index of his intellectual capabilities and thus of the next zone of development of his mind. Of course these capabilities need first of all to be brought to light and established and then drilled and converted from "the next zone of development" (to use Vygotskii's words) into actual mental skills. Teachers frequently are unable to do this in actual practice. But this inability is no reason to adopt the theoretical view that mental development is "independent" of specific content and the actual learning process. On the contrary, one of the basic problems of educational and child psychology is to make this dependency known (and it is by no means direct, simple, or unambiguous).

In this book several aspects of this dependency are analyzed and certain methods of "groping" for it are described. Major attention is devoted to children's mastery and use of knowledge and concepts which seem "unnatural" and "super-difficult" within the framework of traditional understanding of their intellectual capabilities as related to age level. The main task of the research is to study ways of designing academic subjects and children's school work so that much of it becomes "natural" and "accessible" to them.

But it must be definitively stated that such a position is not to be identified with the assertion one meets that "the fundamentals of any subject can be taught in some form at any age" [1:16]. This assertion by J. S. Bruner is of positive value in that he challenges the traditional *absolute* way of characterizing features of the child's mind according to his age level. In itself, the conviction that the child possesses great reserves for intellectual development is correct.

It would not be correct, however, to make these reserves and capabilities absolute. The reason for not making them absolute is not because they are small or because we already see their limits, but rather because we understand the sources and conditions under which the mind and its capacity for cognition are shaped during learning. No matter how strikingly great this capacity for cognition is abstractly, in each particular instance it is the product of many non-psychological factors. First and foremost among such factors are the *social demands* made on the general intellectual development of the person as he participates in a particular historical form of production. In the end social demands are precisely what set the "limits" for the education of the masses and, by the same token, for the extent of their actual mental development as well. The *type of logic* and *means of conveying knowledge* inherent in a particular society also play a large role in intellectual development. Means of conveying knowledge determine the ways stored knowledge and the "norms" for learning at various stages ("age level") of mental development are handed down.

In our view, it is quite important to have specific knowledge of various social sources and conditions of mental development in order to "chart the course" of the intellect at any given age level. Close cooperation with modern sociology, with theory of knowledge, and with logic is indispensable for acquiring such specific knowledge. These disciplines in particular will help overcome the idea, still current among many psychologists and teachers, that the sources and motivation of mental development lie in the individual himself, that is, in his "nature" (sometimes understood to be physical or organic) and in the "inner laws" of his intellectual development (supposedly inherent) independent of the specific historical conditions of his existence and education (in the broad sense of the word).[1]

Sometimes advocates of a "naturalistic" point of view claim they are advocating the "specific character" of psychological principles and are preserving psychological analysis from "flat sociologization." Of course it is necessary to defend the specific character of psychological analysis and to investigate psychological principles proper. But it is important *how* psychological principles are investigated, *how* they are understood, and *in what way* they are found to be connected with the social conditions of human development.

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[1] This is the way the "naturalistic" theory interprets the sources of human intellectual development. It does not refuse to admit the influence of "social" factors on this development, but its "inner laws" are what it makes absolute. A detailed criticism of this theory is given in the work of A. N. Leont'ev [7].

If one knows the particular history and structure of social conditions of human development, then one knows *the very nature of the social individual*. The principles by which the individual appropriates this nature are the very psychological laws by which he becomes a *personality*--a *person*, i.e., *the laws of his psychic development*. Sociology, philosophy, and logic can show how the nature of the social individual is structured and how it functions in society. They do not, however, reveal the specific ways, means, and laws by which *a particular individual appropriates* his socially given nature. This is a matter for *psychology*, but for psychology of the sort through which one can regard man's "inorganic" nature correctly and can turn knowledge of it into a tool for studying the processes of *individual* psychic development.

Karl Marx, in his book bequeathed to psychology said that: ". . . the history of *industry* and the *physical* reality of industry as it has come to be are an *open* book of *essential human forces*, human *psychology* presented to us sensually . . . . The kind of *psychology* to which this book, i.e., sensually the most tangible, accessible part of history, is closed cannot become a really significant or *practicable* science" [8: 594-595]. The "history of industry" is, of course, the maximum and final expression of the "essential forces" of human nature. But the "essential forces" and their derivatives (the structure of material and spiritual culture) are precisely what need to be known by the "practical" psychologist called upon to study the development of the human psyche, as this development occurs in part through socially organized forms of instruction.

In Chapter One,* a specific psychological and psycho-pedagogical analysis of this problem is given (in particular, the correlations between learning and development). Before the specific analysis is given, though, it is important to explain the *general theoretical background* against which psychological research on the formation of the child's intellect, including its features and capabilities as they relate to age level, can be carried out.

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*The next article in this volume (Ed.).

Even though we are firmly convinced that the potential of the human mind is at present difficult even to estimate, it is so great that its determination is mainly a matter of studying and discovering "the secrets of learning."[2] At the same time, it is most important in each period -- including the present -- to study specific, practical (or realizable in the foreseeable future) methods for raising the coefficient of useful learning activity and raising the level of children's intellectual development.

It is impossible to characterize capabilities for learning (particularly with regard to age level) in the abstract. The weakness of Bruner's formulation (the formulation itself but not the actual thought) is that it makes reference to "certain" abstract "forms" of teaching "the fundamentals of any subject" to a child "of any age." The whole problem, really, is to find forms of instruction which are suitable for each specific instance and given age level. And these are not something which can be devised and made "any way you want," as the logic by which these very forms develop is an objective logic determined, as we mentioned above, by many general social factors. If one gropes for the logic behind the development of these forms, the specific phase or step at which the forms and means of learning are changing now, one is really investigating specific new capabilities for learning and their manifestation at various age levels.

At first glance the above statements appear to be only theoretical. But they do, in fact, bear a direct relationship to the methodology and tactics of psycho-pedagogical research into streamlining what is being taught. It is advisable to study the capabilities for learning at various ages by rejecting and departing from the accepted and socially established curriculum (of the elementary school in particular).

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[2] We agree with the writer and scholar I. Efremov in his evaluation of the great capabilities for learning which "the average individual" possesses and the role of genuine "study" in bringing these out, as he has characterized this vividly [2].

intensifying particular sections of curricula, showing that children can learn new content, and showing that the changed curriculum both intensifies further instruction and affects the child's intellect (at first, of course, in experimental conditions) are *practical steps* from our point of view in studying new learning capabilities.

The following facts are therefore important to us. As our experimental investigation shows (see Chapter Two*), as early as the first grade children can be introduced to certain basic *relationships between quantities*, a description of their properties in a system of *formulas using letters*, and ways of using these formulas in the mathematical analysis of one aspect or another of quantities. After these fundamentals have been learned successfully by "ordinary" first-grade children, the concept of number as a form of representing *a particular relationship* between quantities may be introduced, a very farsighted move for the further study of numbers (fractions, in particular). Moreover, first-graders are capable of handling such a problem as follows. Given the formulas

$$\frac{A}{n} = 4 \quad \text{and} \quad \frac{A}{m} = 6,$$

children can find that $n > m$. That is, they are guided by the *complex dependencies* existing among the objective facts of dimension, measure, and number.

The work reported in this book is focussed on a greater broadening and deepening of intellectual skills than is traditionally called for, capabilities of the child's mind which are not ordinarily taken into account nor especially "cultivated." Second-graders who follow the experimental Russian language curriculum (see Chapter Three) are able systematically to isolate, analyze, and describe the grammatical forms of an *artificial language* on their own which is concrete evidence that these children have begun to think about *the complex innerrelations* between the form of a word and what it communicates. If this is the kind of knowledge that the child is able to begin to ponder, then the whole subsequent course of study of his native language can be made more interesting, more serious, and, most important, more intellectually challenging than with traditional grammar.

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*The last three articles in this volume (Ed.).

The material presented in this book shows that the intellectual capacity of children in the primary grades is considerably more extensive and more varied than that toward which the accepted, traditional content of elementary instruction is oriented. We believe this to be a proven fact [3,5].

The results were obtained while teaching *experimental curricula* especially designed to ascertain certain "broader" intellectual potential in primary school children. Therefore, it is not claimed either that they are complete or that they can be instituted in schools on a mass scale in the form in which they were devised for experimental purposes. We are, in fact, becoming more and more convinced that if the curricula are elaborated somewhat -- primarily in teaching methods -- they are ready to be tested on a much broader scale than the demands of "pure" psychological experimentation dictate.

Devising experimental curricula (in mathematics, Russian language, and manual training) and using them in teaching are thus the particular *method* used to investigate capabilities for learning based on age level [5]. Of course, as such investigations are carried out, a number of issues are encountered concerning the general ways of structuring academic subjects which in turn leads to the necessity for considering certain specific methodological problems. And the latter cannot be analyzed without "involvement" in the subject matter -- mathematics, linguistics, and so forth.

Academic subjects actually need to be structured *jointly* by representatives of various disciplines. But since such teamwork has not yet been worked out as well as it should, psychologists by necessity must make excursions into the various disciplines. True, only what pertains directly to setting up an *academic subject* interests the psychologists. The main issues concern the nature of abstraction, the genetic connection between concepts, means of expressing concepts in symbols, and so forth. But if the fundamentals of the experimental curricula are not explained in detail it would be difficult -- at the present stage of work, in any case -- to describe the actual instruction process which leads to the discovery of new capabilities for learning. Considerable space, then, is devoted to an analysis of mathematical and linguistic issues (but only from the standpoint of *educational psychology*).

For a particular purpose, research into one aspect of the so-called "formal intellect" -- the internal (mental) level of operations (see Chapter Five)-- was considered in our work. But the data obtained are significant only in the context of a total investigation of the child's intellectual capabilities.

As said before, designing experimental curricula was our method of working. One must keep in mind, however, that setting up and providing for such work is exceedingly complicated, time-consuming, and crucial because it has to do with the actual learning process in real schools. Much of the burden here falls on those who actually implement all the experimental notions -- the teachers and school administrators. These people have been a constant source of help, support, and businesslike critical appraisals of all our work which are so necessary in a new and complicated matter. Our research group takes this opportunity to express its sincere appreciation to all the teachers who used our curricula and to the administration of Schools No. 91 and 786 in Moscow, School No. 11 in Tula, and the village school at Mednoe, Kalinin province, for their help in organizing the research.

This book is arranged as follows.* The introduction (by D. B. El'konin and V. V. Davydov) and Chapter One (by El'konin) state the theoretical bases for the experimental work and define our approach to the study of primary school children's capabilities for learning based on their age level. Chapter Two (sections one and two by Davydov, and section three by G. I. Minskaya) characterizes the capabilities as they apply to the study of mathematics in first grade. Chapter Three (section one by L. I. Aidarova and section two by A. K. Markova) deals with the capabilities as they apply to the study of the Russian language in grades two through four. Chapter Four (by E. A. Faraponova) deals with the capabilities as they apply to manual training in first grade. Chapter Five (by Ya. A. Ponomarev) contains material showing the connection between instruction according to experimental curricula and aspects of the internal level of operations.

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*Only Chapters 1 and 2 of the original work are reprinted in this volume (Ed.).

We have been able to set forth only some of the material we have obtained since the first collection of our papers [4] came out. The research is continuing, and new problems and tasks are appearing. Still, it is hoped that by reading this book educational theorists, methodologists, child psychologists, and thoughtful teachers will be convinced that the untapped capabilities of children are great and that much remains to be done to improve school instruction, what the children are learning, and the tempo and level of their mental development.

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PRIMARY SCHOOLCHILDREN'S INTELLECTUAL CAPABILITIES AND THE CONTENT OF INSTRUCTION*

D. B. El'konin

The advent of universal compulsory eighth-grade education and, in the near future, tenth-grade education in our country broadens the opportunities for significantly improving the structure, content, and methods of academic instruction, which, in turn, is involved with our ever-growing interest in problems of the child's mental and personal development. In order to benefit from these opportunities and to avoid haphazard and premature solutions to the practical problems facing public education, a number of complex scientific issues need to be worked out ahead of time. Important conditions for a scientific approach to these tasks is an improvement in the level of theoretical and experimental investigation into the mental development of schoolchildren and, in particular, increased attention to an analysis of the theoretical views in Soviet education and psychology on these problems. At the same time, of course, a special examination of the *history* of the connection between instruction and pupils' mental development needs to be done.

The problem of this connection confronted child psychology as far back as the 1930's, when a significant change occurred in the educational system -- the transition from *comprehensive* instruction to instruction *by subjects*. Thirty years ago two books devoted to issues in the mental development of schoolchildren came out, each written by an outstanding Soviet psychologist -- P. P. Blonskii [2] and L. S. Vygotskii [8].

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*From *Learning Capacity and Age Level: Primary Grades*, edited by D. B. El'konin and V. V. Davydov, Moscow, Prosveshchenie, 1966, pp. 13-53. Translated by Anne Bigelow.

Blonskii, relying on analysis of curriculum content, attempted to characterize the thought process at each stage in the school years and the conditions for the child's transition from one form of thinking to another. He wrote:

> In teaching children, the school inevitably must consider the extent to which their thinking is developed. We may therefore confidently assume that to some extent curricula reflect the general course of development of the pupils' thinking.
>
> Rather than analyzing any specific curriculum, it would be more expedient to take the content that the most authoritative curricula all have in common and that to which there are no weighty objections from anyone. On this basis we can assume that the part of the curriculum on which the teachers completely agree actually gives a true picture of the development of a child's thinking.
>
> But it does so, of course, only in its general, approximate features, and from these curricula we can hope to obtain only the most general picture of the development of the child's thought process, satisfactory only at the beginning of the investigation, as a point of departure for it [2:158].

Blonskii divided the school years into three stages: early prepuberal childhood (ages 7 to 10), late prepuberal childhood (ages 10 to 12 or 13), and pubescence (ages 13 to 16).[1] As a summation of his curriculum analysis Blonskii outlined the general course of development of the thought process as follows. Early and late childhood is characterized by thinking according to rules and by striving for detail; pubescence is characterized by proof-seeking, including skill in mental detail. Early prepuberal childhood is the period of concrete thinking, late prepuberal childhood is the period of thinking in relationships, and pubescence is the period of abstract thinking [2:169-170].

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[1] We shall not consider the principles Blonskii used to divide the school years into periods.

The method of analysis which Blonskii used may be exemplified most clearly by his discussion of the mathematics curriculum.

> In almost all the curricula, arithmetic is to be studied in the lower grades, and algebra in early adolescence. The psychological difference between arithmetic and algebra is that in the former, when one operates with numbers (figures), one is thinking in particular empirical numbers, whereas in the latter, when one operates with letters, one takes them to mean any numbers of a given type. Abstract thought reaches its culmination in algebra: In it, thought is abstracted even from empirical numbers.
>
> By analyzing mathematics curricula one can chart the important landmarks in the development of abstract thought in schoolchildren in so far as the study of such a maximally abstract subject as mathematics is a good indicator of the maximum level which children's abstract thought will reach at various ages. Arithmetic and algebra -- first being where the qualitative distinctions of objects are abstracted, so that only the fact that they are objects, that is, only their distinctness (only number) remains, and the second being where even the specific numerical values of objects are abstracted -- these are the two basic stages.
>
> The school arithmetic course breaks down clearly into two parts -- whole numbers and fractions, where concrete numbers usually form the transition from the first part to the second. Whole numbers are studied in younger (ages 7 to 10), and fractions in older prepuberal childhood. Through the study of whole numbers in younger prepuberal childhood, the child will reach the stage at which the qualitative attributes of an object are abstracted, the stage of quantity and value. Through the study of fractions the child will reach a second stage -- the stage of quantitative relationship. This latter stage is the stage of abstract thinking about the relations of objects devoid of all qualities. The stage of thinking in abstract relations thus follows the stage of thinking in qualitative abstraction [2:161-162].

The thinking process develops further in early adolescence as the child studies algebra, and in particular, as he solves equations. Blonskii wrote:

> At this age the child learns to operate with abstract general numbers instead of specific empirical numbers, and to establish maximally general and abstract laws about numbers. This, after all, is essentially what constitutes the unit on proportion and the solution of equations based on it [2:162].

Summarizing his analysis of the development of the thought process in the study of mathematics, Blonskii commented that:

> The fundamental stages of development of asbtract thought in pupils can be perceived in mathematics curricula. They are: 1) the stage where qualitative attributes of objects are abstracted so that only particular empirical numbers and relations between numbers remain; 2) the stage of general abstract numbers; and 3) the stage of abstract quantitative laws [2:163].

The particular research which Blonskii and his associates interpreted in the book we are discussing[2] basically corroborated the general picture of the development of children's thinking they obtained through curriculum analysis. Of course, the research added detail to the general picture, intensified it, and posed a number of problems concerning the connection between the development of thinking and the development of perception, memory, and speech.

In his final chapter, Blonskii returned to a general outline of the development of the thought process and examined it as it relates to improvement in perception and memory, but this time from a genetic standpoint.

> Early prepuberal childhood is the age of very intensive development of purposive attention, and late prepuberal childhood is the age in which the mnemonic function achieves its maximum development, while adolescence is the age of problems, reasoning, and arguments. The function which is maturing at the greatest rate--thinking--begins to manifest itself with great energy, and it plays a tremendous role in the life of an adolescent and young adult [2:278]. (italics ours--D. E.).

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[2] See Chapter III, "The Development of Concepts at the Grade School Level"; Chapter IV, "Understanding, Keenness of Observation, and Explanation at the Grade School Level"; Chapter V, "Learning and Thought"; and Chapter VI, "Rational Thinking" [2] (it is not our task to analyze the methodology of this research or the results --D. E.).

> Thinking is one of the functions which in both ontogenesis and phylogenesis develops later than many others. Because we do not wish to disparage or underrate children, we should not disparage the thinking process by ascribing an unnecessarily great capacity for it almost to babes in arms. The thought process develops on the basis of the most elementary intellectual functions, and in order to be capable of reasoning, one must already be observant, have sufficient practical experience and knowledge, and possess sufficiently developed speech [2:279].

Blonskii thus finds that perfection of the most elementary functions of perception and memory is prerequisite to the development of the ability to think. This thesis is important, because it necessitates a return to the principles for designing curricula.

Actually, since teaching is supposed to assist intellectual development, and since the development of the ability to think at the early stages of school is determined by the development of perception, then it is natural that the use of visual methods -- not only as a didactic principle but also as a principle for the selection of material -- should be basic to designing curricula for the primary grades. Blonskii considered it proper that "curricula for the early grades are constantly emphasizing the development of the child's powers of observation, while methodology reiterates the importance of visual methods in teaching primary school children" [2:275].

His approach to the development of memory was analogous. Because memory develops most intensively in older prepuberal childhood (ages 9 to 12) and verbal memory attains its maximum development at this age, then "it is understandable why the memorizing of poems, of the multiplication tables (and of tables of addition, subtraction, and division, in fact), and of all kinds of rules, geographical names, and so forth, occurs in the primary and middle grades" [2:245].

Thus Blonskii first analyzed the curricula and inferred from them the general developmental characteristics of the child's thinking process, and then proceeding from more detailed research, substantiated the content of these very curricula by making references to the "characteristics" of the development of the thinking process.

As a result it appears to be correct to assert that these curricula correspond to the logic of the child's mental development and, moreover, that they have been substantiated by psychology. Naturally, while these curricula might be improved in details, they cannot be altered in any substantial way, for this would contradict the laws of the child's mental development.

Thirty years have passed since Blonskii's book came out. A great deal of research has been done since then on the child's thought processes. All of it has basically corroborated the characterization of the thought process made by Blonskii through analyzing curriculum content and the logic of its design. It is interesting that he acquired his data by investigating concepts which were formed outside the formal instruction process as well as those which were a part of what was being taught.

Blonskii's study has been discussed in detail because it demonstrates most sharply flaws of research into the mental development of children and because it permits a number of questions to be posed about children's mental development. But first the origins of the conceptions of children's mental development to which the curricula are oriented must be looked into, as well as the principles by which the curricula were designed.

Formal instruction, schools, and curricula have a very long history -- considerably longer than that of scientific child psychology. Scientific child psychology appeared in the nineteenth century, while instruction reaches far back in time. Of course, teachers and curriculum designers in the past have had certain empirical notions about the child's mind. But what determined the choice of curriculum content was not so much empirical notions as the tasks society demanded of the training and instruction of the younger generation. In a class society these tasks were different for children of each social class -- for children of peasants, workers, tradesmen, landowners, and capitalists.

One needs but to recall the school system in prerevolutionary Russia (the parish school, the district elementary school, the city elementary school, the city academy, the progymnasium, and the gymnasium) to clearly imagine the degree of *differentiation* of instruction which then existed depending on the aims of the ruling classes in the realm of education and to imagine the limitation on the tasks for instruction in the schools for those at the "bottom."

The system of differentiated instruction, intended for various classes (and layers) of society, was structured according to the principle of *closed concenters* of knowledge and skills. Historically, concentrism as a principle for organizing curricular material grew, in our view, out of a mechanical process of arranging the types of schools one on top of another. There were four such closed concenters in Russia's prerevolutionary schools: (a) the *grammar school* in which only the skills of reading and writing were taught (chiefly Church Slavonic); (b) the so-called *public elementary school*, in which practical skills in reading, writing, and counting, and a range of elementary information about natural phenomena were provided; (c) the *city academy* and the *progymnasium*, in which a summation of empirical knowledge from various disciplines (geography, history, natural sciences) was provided; and finally (d) the *gymnasium* (or a comparable educational institution), in which a strictly theoretical education was provided (in the classical humanities or the sciences).

Certain of these closed concenters were dead ends of a sort. Transferring from a lower type of school to a higher one was impeded not only by direct political and economic obstacles but also by the limited instruction in the lower concenters as compared with the higher ones.

Although historians of the schools (but not just historians of educational ideas) need to analyze the particular types of schools and the historical conditions which determined their rise, the only historical fact of importance here is that the elementary school began long before children's mental development was studied scientifically. The content of instruction in the elementary school was dictated first and foremost by the tasks set for it by the government, which was the mouthpiece for the capitalistic society.

A striking example of how the ruling class in the person of the tsarist government limited changes in the content of elementary instruction in every possible way is the debate over the books by K. D. Ushinskii, *Our Mother Tongue* (Parts I and II), which appeared in 1864, soon after the emancipation of the serfs. Ushinskii championed the idea of *instruction as development*. He understood that mental development is organically connected with the content of instruction. However, even the very limited and, it would seem, politically neutral changes Ushinskii tried to introduce aroused stormy protest among the bureaucrats of the tsarist government.

Even the philosophers and teachers who were the most progressive of their time (Comenius, Rousseau, Diesterweg, Ushinskii, and others) were always limited in their attempts to formulate principles of instruction (didactics) by the tasks the ruling classes of society set for the school. It might be asked whether the didactic principles formulated by teachers of the past are an ideological expression, of sorts, of the limits which society placed on the education of the masses. We are inclined to think that they are. If some of these principles are examined, such as *the use of visual aids*, the principle of *comprehensibility*, and others, it can readily be seen that each contains both *progressive* and *conservative* elements. The progressive element was aimed against the scholastic school and against the idea that knowledge is incomprehensible to the mass of children and the conservative element was aimed at limiting the content of education. The progressive element of the principles had significance at the very beginning of the struggle within bourgeois society for the education of the children of the masses. It was proof that such education was possible, and it showed the conditions under which it could take place. But as the schools developed, the significance of the progressive element dwindled, while that of the conservative element, which limited the content of instruction, increased. Still, the didactic principles governing teaching technique and the selection of content, at the time the public elementary school was being set up, were an expression of the curriculum designers' views about the processes of mental development and their relationship to teaching. Even the most progressive teachers from the seventeenth to the nineteenth centuries (at least before the time of Marx, and the majority until this century) viewed the mental development of the child as a maturing process. This naturalistic approach to mental development gained even more support after Darwin. Development was viewed as gradual maturation, a natural process which follows the inner logic of natural laws in the same way as embryonic development does. This point of view is still held by certain foreign investigators of children's mental development.

Quite naturally, so long as this view of mental development prevailed, instruction was only able to follow this naturally unfolding development and make use only of its finished products. Vygotskii described this approach in the following way:

> The view of the relationship between instruction and development held first and most widely here up to now has been that instruction and development are thought of as two independent processes. Child development is represented as a process which obeys natural laws and is a kind of maturation, while instruction is understood as a purely external use of the opportunities which arise in the developmental process [9:251-252].[3]

One thing is certain. During the struggle to establish the public elementary school in Russia in the mid-nineteenth century, the point of view that curriculum designers took was naturalistic. Like any curricula, the elementary school curricula of that time embodied materially the tasks assigned to instruction and the views of mental development which their designers took. Didactic principles are only their concrete expression.

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[3] Note that Vygotskii indicates that this viewpoint was very widely held here, that is, in Soviet education and child psychology of the thirties. Unfortunately he does not name specific proponents of these views. Perhaps he was unable to because the view was so widely held.

We know that elementary school curricula have undergone no essential changes up to the present. The Soviet mathematician and teacher A. Ya. Khinchin, indicating that our mathematics curriculum is a poor copy of prerevolutionary curricula, wrote: "In our country, where every worker is a conscientious participant in production, *school mathematics should not be restricted* to the bourgeois inculcation of bare recipes and narrow practical skills which open no scientific perspectives" [5:19] (italics our -- D. E.).

The same is true of the Russian language curriculum and of others as well, even though the elementary school underwent *radical reforms* immediately after the October revolution. It is not simply that elementary education became universal and compulsory. Fundamentally, the school was assigned new tasks, and the content of instruction was reexamined. Scientific materialistic knowledge about nature and society penetrated thinking concerning the elementary school, a reformation which resulted in a radical difference between the post-revolutionary elementary school and the prerevolutionary, "public" elementary school.

But in spite of the fundamental change in the content of elementary instruction, the ways in which it is organized have remained as they were before. The elementary school has remained a closed concenter of skills and elementary ideas and the content (even though it is new) has been organized on the basis of didactic principles which limit young pupils' opportunities -- such principles as *the use of visual aids*, *concreteness*, and *comprehensibility*.

The causes of this situation, while varied, are primarily historical and socio-economic. While it is the task of historians of the Soviet school to analyze such causes, one cause is of interest and is indicated. The basic approach to mental development and instruction was carried over along with the curricula, which are new in content but old in the principles of their design.

One begins to understand why Vygotskii regarded the view that instruction is a purely external use of opportunities arising in the developmental (maturation) process as the one most widely held views. However, not only did Blonskii justify the existing elementary curricula; he substantiated it with data from psychology. At the same time, he pointed repeatedly to the influence of the school on mental development.

> My varied investigations, described in the preceding chapters, have convinced me that the schools' influence on the thinking process, beginning from the day the child starts school, becomes particularly clear in adolescence. . . . In particular, the enormous influence of such a public institution as the school on the development of the thought process has become evident in all of our research, in both particular and general conclusions" [2 :281-282].

Blonskii, in recognizing the influence of the school on mental development, was guarded. The reason the school's influence on the development of the thinking process "becomes particularly clear in adolescence" has been answered by certain researchers. As they see it, the maturation process is coming to an end by adolescence and, for this very reason, instruction begins to exert its decisive influence. But instruction is not influencing development in the proper sense of the word, nor is it influencing the appearance of new forms of mental activity as these forms have already developed fully by this time. Instruction exerts its influence not on the appearance or initiation of forms of mental activity, but _only on the level to which they are developed_--it only _exercises_ them.

Vygotskii noted this connection between the theory that mental development is maturation and the view that instruction is exercise when he wrote that:

> The child's memory, attention span, and thinking process have developed to the degree that he can be taught reading, writing, and arithmetic; but if we teach him these, will his memory, attention span, and thinking process change or not? The old psychology answered the question this way: They will change to the extent that we exercise them, but nothing in the course of their development will change. Nothing new has occurred in the child's mental development because we have taught him to read and write. He will be the same

> child, only literate. This view, which epitomizes the old educational psychology, including Mieman's famous work, is brought to its logical limit in Piaget's theory [9:253-254].[4]

Blonskii's conclusion that instruction exerts its decisive influence on the development of the thinking process only in adolescence indicates a very significant defect in the curricula and in all the content of instruction in the elementary school. Curricula are oriented toward the already developed facets of the child's mental activity and provide nothing but practice material for them. The theory that mental development is maturation logically necessitates the theory of instruction as exercise. It is no coincidence, therefore, that one of the central methodological problems has long been that of exercises -- how many, what kind, how fast to increase their difficulty, and so forth.

At the same time, Blonskii's observation concerning school instruction and maturation raises another issue as well. Why does the school not exert an influence on mental development in the elementary grades as it does at the adolescent stage? This issue has been noted in more recent investigations. B. G. Anan'ev, who made a special study of elementary school instruction, came to the following conclusion:

> In comparison with the other stages in elementary instruction, the greatest advance in the child's development actually occurs in the first year of instruction. After this the rate of mental growth slows down somewhat, as a result of insufficient attention to the developmental aspect of instruction. Paradoxical phenomena appear: As the sum of knowledge and skills acquired increases, the child's mental powers and capabilities, especially for generalization and practical application of this knowledge, increase relatively more slowly. Progress through the material the child is taught does not bring an automatic increase in what he can be taught. This phenomenon deserves careful study, inasmuch as it is evidence that many possible educational influences on child development, on the formation of the child's personality, and on his endowments have not been used in actual elementary instruction, and the inconsistencies between instruction and development have not been fully overcome [1:24].

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[4] We cannot agree with Vygotskii that this is an old view. After all, it is being developed by the contemporary psychologist Piaget, it is presented in Blonskii's work, and it persists in curricula. It is "old" in the sense of when it was originated, but unfortunately it is not yet out of date.

L. V. Zankov has noted the insufficient influence of elementary instruction on mental development:

> Our observations and special investigations in schools in Moscow and the outlying districts will testify that no particular success in pupil development accompanies the attainment of a difficult level of knowledge and skills in the early grades [11:20].

Thus, the contemporary investigators Anan'ev and Zankov are obtaining the same results as Blonskii did thirty years ago. This indicates first, that in these thirty years, no essential changes have occurred in elementary instruction, and second, that instruction does not have enough influence on the mental development of children in the elementary school. In light of these facts it is not at all surprising that when they enter secondary school, the children turn out to be insufficiently prepared to master systematic courses, such as mathematics, languages, science, or history, the evidence of which is a decline in good grades.

Why, in fact, doesn't elementary instruction exert necessary and sufficient influence on the child's mental development? Vygotskii's views, as spelled out in a book [8] which appeared at the same time as did Blonskii's, are of interest here.[5]

There is a difference between Vygotskii's and Blonskii's characterization of mental development for school age children. Vygotskii thought that mental development was characterized not so much by the level of development of specific mental processes as by interfunctional connections and their changes. As he saw it, each period of mental development involved a certain structure of the mental processes, with the function developing most intensively at the period located at the center and influencing the total mental development. Vygotskii wrote that:

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[5] Vygotskii's basic works were reprinted rather recently and are widely known [9, 10].

> The child's psychological development consists less in the development and perfection of particular functions than in a change in interfunctional connections and relations, on which the development of each partial psychological function in fact depends. Consciousness develops as a unit, changing its internal structure and the connection among its parts at each new stage, and not as the sum of the partial changes taking place in the development of each specific function. In the development of consciousness, what happens to each functional part depends on a change in the whole, not vice versa [9:242].
>
> Everything we know about mental development teaches us that interfunctional relationships are neither constant, insignificant nor capable of being removed from the brackets within which the psychological calculation is being performed, but that a change in the interfunctional connections, *that is, a change in the functional structure of consciousness, in fact constitutes the chief and central content of the entire process of mental development as a whole* [9:243-244].

His characterization of the specific periods of mental development is related to the above interpretation of the process of mental development.

> What we know about the child's mental development of consciousness, characterized by a lack of differentiation of specific functions, is followed by two other stages -- early childhood and the preschool age. In the former, *perception* is differentiated, goes through its basic development, is dominant in the system of interfunctional relations, and (as the central function) determines the activity and development of all the rest of consciousness. In the latter stage, *memory* comes to the fore as the dominant central function. Perception and memory, then, have matured considerably by the time the child enters school and are among the fundamental prerequisites for total mental development at this age [9:244].

The *development of the intellect* comes to the fore in the early school years. This development is, in fact, what leads to a qualitative reorganization of perception and memory (which developed earlier) and to their conversion to *purposive processes*. Vygotskii explained this statement in one of his last lectures on mental development at the school age.[6]

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[6] He delivered it on February 23, 1934. We have a shorthand record of it, certain parts of which we can cite here (unfortunately, the text of this record has not yet been published).

> This, of course, must be understood very conditionally. First it is necessary to emphasize the words "is developing," and not "has developed." This does not mean that the child is a thinking creature by the time he goes to school; it means that his intellect is functioning quite weakly at this stage. One might say that the school child [at the beginning of this age] has a pygmy intellect with grandiose potentialities for remembering and even more gradiose potentialities for perceiving. Consequently, the intellect is not a powerful and predominant aspect of mental activity at the very beginning; on the contrary, it is at first exceedingly weak in comparison with the functions that have matured during the earlier stages. But it goes through its maximum development during the [early] school years, unlike memory and perception at this time.
>
> If we compare the original and the final states of the intellect at the school age and the original and the final states of the memory and the attention span, it turns out that the original and the final states of the intellect will be widely divergent, while the original and the final states of the memory and the attention span will diverge little; that is, the intellect is moving into the center of development.

To Vygotskii, the consequence is

> . . . that each of these functions [perception and memory] in turn becomes intellectualized, that is, they change as they are penetrated by the components of intellectual activity. . . . This means that these functions become more and more closely coordinated with the intellectual operations, that they have favorable conditions for their development, and that they advance and develop insofar as they are a part of what is fundamentally developing at this age.[7]

Thus, in Vygotskii's view, changes in memory and perception during the early school years are secondary, a consequence of the development of the intellect. When describing the development of the intellect itself, Vygotskii said in the same lecture:

> The new form of inner activity at the school stage consists of the following: While at the preschool stage these inner activities are directly connected with outward activity, at the school stage we have inner activities which occur relatively independently of outward activity. Now we have a child . . . in whom inner and outward activity is being differentiated.

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[7] For a more detailed discussion of the problem of the intellectualization of functions and of how the child becomes conscious of them and makes them subject to his will, see the works of Vygotskii [9, 10].

Vygotskii thought that the early school age was actually the period in which the thinking process develops actively. This development consists primarily of inner intellectual activity independent of outward activity, a system of strictly mental activities. Perception and memory develop under the determining influence of the intellectual processes that are taking shape.

Blonskii presented mental development somewhat differently. "The first half of prepuberal childhood," he wrote: "is the age of the fastest, most intensive development of so-called purposive attention. But what psychologists usually call purposive attention is none other than perception regulated by thought" [2:276].

He continued:

> Late prepuberal childhood is the age in which the mnemonic function achieves its maximum development, and adolescence is the age of problems, reasoning, and debate. The function which is maturing at the greatest rate — thinking — begins to manifest itself with great energy, and plays a decisive role in the life of an adolescent [2:278].

As if he were "frightened" of this overly high estimation of the development of the thinking process in adolescence, Blonskii added:

> And yet although this thinking process reflects relationships within the objective material world and demonstrates a certain awareness of time, it is still lacking. While it is comfortable enough in the concrete world, abstract thought is still a long way from being fully mature. An intensive development of abstract concepts is only beginning in early adolescence and continues with greater intensity in later adolescence. . . . Abstraction -- thinking involving generalization -- can develop to the proper extent only on a foundation of rich concrete material, that is, abstract thought can develop only when concrete thought is highly developed [2:278-279].

In Blonskii's periodization, early prepuberal childhood extends from the ages of seven to ten, late prepuberal childhood, from ten to twelve, and pubescence, from thirteen on.

If the way in which these two authors characterized mental development in the early school years is examined (from the ages of seven to eleven or twelve), certain differences can easily be noted. In Vygotskii's view, by the beginning of the early school years perception and memory have gone through the bulk of their development. Blonskii believed that this period is characterized by the intensive development of perception during the first half and of the mnemonic function (memory) during the second half. Adolescence, to Blonskii, is the period of the most intensive development of the thought process but to Vygotskii the most intensive development of the thought processes is in the early school years.

These two investigators describe the same period of mental development. Moreover, the facts both of them cite are accurate on the whole. The problem is not that one's results contradict the other's. Of course, each had at his disposal an arsenal of data which differed from the data of the other. But each was adequately informed of the other's work, and knew and took into consideration his data. At issue then is not their facts but instead their fundamentally different approaches to singling out characteristics of mental development at any given period. At least two approaches are possible. The first is to single out the aspects of mental development which are concluding their formation during a period. Of course, both memory and perception are developing during the early school years. While they do not stop developing they acquire a relatively finished form during these years (on which both investigators agree). If the view is adopted that, at any given period, mental development is characterized by the processes which have gone through the bulk of their development and are just being completed, then Blonskii's characterization would be correct. But another approach is possible. The aspects of mental development which are being differentiated for the first time and are only beginning their relatively independent and intensive development can be isolated. If this view is adopted, then Vygotskii is correct in his characterization of the primary school pupil.

The central issue, therefore, is whether to characterize a given age level according to the processes which are concluding their development or according to those which are only beginning. This issue is important not only for child psychology but for orienting actual teaching as well.

It is clear, then, why Blonskii considered it proper that "curricula for the early grades are constantly emphasizing the development of the child's powers of observation, while methodology reiterates the importance of visual methods in teaching primary school children " [2:275]. Elsewhere, in discussing the role of thought and memory in learning, he wrote:

> The most accurate pedagogical conclusion to be drawn from what has been said in this chapter would be that the child is basically occupied with learning by thinking, and the basic function at this age is remembering by thinking, that is, memorization accompanied by pondering what and how to memorize, and recall accompanied by pondering what and when to recall [2:97].

Instruction, its content, and the methods organically connected with it should thus be oriented toward the development of observation and verbal memory as the bases for the future development of the thinking process, that is, oriented toward the processes which are either almost completely developed or are already developed. From this point of view the curricula being followed, then as well as now, are fully justified.

Vygotskii approached the problem in a fundamentally different way. It is well known that he especially emphasized the key role of instruction in mental development. In itself this thesis is not new, having been put forth by many progressive teachers of the past, such as Ushinskii. But the modern discussion of the problem is the work of Vygotskii. It is interesting to recall his attitude toward the preschool and school instruction of his day:

> Teaching should be oriented toward the child's future, not his past development. Only then will it be able to unlock the processes of development that lie in the area of immediate development.

> Let us clarify this by a simple example. As we know, the comprehensive system of academic instruction was given "pedagogical substantiation" while it prevailed here. Teachers affirmed that it corresponded to the characteristics of the child's thought processes. The basic error was that the matter was stated wrongly in principle -- a result of the view that instruction should be oriented toward past development, toward the aspects of the child's thought process which had already matured. Using the comprehensive system, teachers proposed to consolidate what the child in his development should have left behind when he started school. They oriented themselves toward what the child could think out on his own and did not consider the possibility of his shifting from what he could do to what he could not. They evaluated the fruit that was already ripe. They neglected to consider that instruction should carry development forward. They did not take the next area of development into consideration. They were oriented toward the line of least resistance, toward the child's weak side rather than his strength.
>
> The situation becomes reversed when we begin to understand the reason why the child entering school with functions which matured at the preschool stage tends toward thought patterns that correspond to the comprehensive system. The comprehensive system is none other than the transfer to the school of a system of instruction adapted to the preschooler -- the consolidation, during the first four years of school, of the weak aspects of preschool thinking. This system lags behind the child's development instead of leading it [9:277-278].

Recently the relationship between instruction and development has again attracted the attention of scholars. Several years ago Zankov and a group of collaborators began a special investigation of the interrelations between learning and development [11, 12]. The material published does not yet give us a chance to judge the progress they have made toward solving this problem. The concept they are developing will have to be critically analyzed in detail sometime in the future. Meanwhile several purely theoretical remarks are made.

First, Zankov interprets certain of Vygotskii's theses in a very odd way:

> The substance of Vygotskii's theoretical views is that the development of the child's mental activity is genuinely social in nature. Collaboration and instruction are the determining conditions of it. At the same time, development is not to be equated with mastery of knowledge and skills: Mental functions are restructured and take on a new character during instruction.
>
> This approach to the problem is very important both theoretically and practically. It correctly orients educational theory and practice in that is stimulates the creation and application of teaching methods that are highly effective in promoting pupils' mental development.
>
> In setting up the learning process in our experimental class we are proceeding from Vygotskii's theoretical views and are structuring this process so that instruction precedes development and thus achieves the optimum results in the development of the intellect [12:12].

One important correction needs to be made in this interpretation of Vygotskii's views. Nowhere did Vygotskii relate the high effectiveness of instruction for development to the *means* of instruction. On the contrary, he always emphasized the *content* of what was being learned as having primary significance for mental development. We know that he approached the problem of instruction and development as it related to the problem of "worldly" and scientific concepts, which for him was a *model* of the relationship between instruction and development. He thus wrote, in fact, "Essentially *the problem of non-spontaneous and, in particular, scientific concepts is a problem of instruction and development* . . . " [9:251]. In addition he indicated that "from the standpoint of logic, the differentiation of the spontaneous and non-spontaneous concepts children form coincides with *the differentiation of empirical and scientific concepts*" [9:250] (italics ours -- D. E.).

Vygotskii's basic idea was that the greatest strides in developing the intellect during the school age -- becoming aware of mental processes and mastering them -- are made "*through the gateway of scientific concepts*" [9:247]. He thus thought that decisive progress in development was associated primarily with the content of instruction. As a result, he was less concerned with working out and applying "effective teaching methods" than with the content of instruction, the scientific character of content in particular (the "methods" themselves are derived from the content).

To amplify Vygotskii's views a bit, it may be said that if the basic content of school instruction remains empirical knowledge, then no matter how stimulating and effective the methods of instruction are, this content does not become the determinant of the pupil's basic mental development. Instruction in this case only exercises and thereby improves the mental processes which are involved in the mastery of empirical knowledge, the development of which is characteristic of the preceding period of development. Although he did not investigate it fully, Vygotskii foresaw the organic connection between mental development during the school years and mastery of scientific concepts specifically.

A second misunderstanding has to do with Zankov's interpretation of Vygotskii's statements concerning "the next area of development." Zankov wrote:

> When we analyzed the factual material we had obtained in our research we came to the conclusion that the so-called next area of development is not (as Vygotskii assumed) the only way for instruction to influence child development.
>
> The specific role of instruction is manifested not only when the adult is using leading questions and examples to help the child in his intellectual activity and the child is imitating the adult. The teacher can organize the material the child is using in a definite way so the teacher is not helping the child but is letting him perform the tasks wholly on his own. Imitation is thus completely excluded. Meanwhile, as he solves the problems on his own, the child is progressing in that particular area of mental activity [12:12-13].

Then Zankov cited an instance in which children, on their own, examined in succession three objects which had much in common. Observation improved with each object, and the children noticed twice as many characteristics in the third as in the first. The sense in which the

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teacher "did not help" the children is not clear in this example. He did not in fact show them how to perform the operation, but he did show what kind of operation was needed. Although the instructions mentioned description, the children in fact were _comparing_ the objects. They performed this operation on their own to a certain extent, but _it was not new_ to them. There can be no doubt that the first-graders had already had practice in making such comparisons and had been taught to do this.

Thus, in this example, perfection of an operation in conditions which change while the problem is being examined are being dealt with. Zankov came to far-reaching conclusions on this basis, however.

> The facts cited provide a basis for assuming that instruction influences pupils' development _in various ways_. These ways are not isolated from each other, of course, but are in complex interaction. The formation of so-called "next area of development," in particular, interacts with other ways in which instruction influences the development of pupils' mental activity.
>
> One of the important tasks of research is to ascertain the varied types of relationship between instruction and development and to study the interaction among these types. It is of great importance for theory to perform these tasks, since it will lead to a fuller knowledge of the interrelations between instruction and development.
>
> One should not underestimate the importance of the solution of these tasks in actual teaching either. The formation of a "next area of development" as a definite way of influencing mental development is characterized by the teacher's showing how to perform a task and the pupil's imitation of him. The types of relationship between instruction and development in which the emphasis is shifted to pupils' independent intellectual activity are limited considerably here, as a consequence. And yet this very approach to structuring the learning process is more important now than every before [12:15-16].

We have cited this statement by Zankov because it demonstrates quite clearly the principal distinction between his approach to the problem and that of Vygotskii. Vygotskii did in fact believe that the influence of instruction on development is determined to a great extent by the guiding role of the adult in instruction. Thus he wrote:

> An animal, even the most intelligent one, is incapable of developing its intellectual capabilities through imitation or instruction. It cannot master anything fundamentally different from what it already has. It is capable only of being trained. In this sense it can be said that the animal is not capable of being taught at all, if we take "teaching" in a specifically human sense.
>
> For the child, on the other hand, development through instruction is basic -- development through collaboration involving imitation, the source of all the specifically human mental attributes. Thus the opportunity, through collaboration, for rising to a higher level of intellectual potential, the opportunity for moving with the aid of imitation from what the child can do to what he cannot -- this is the key to all of educational psychology. The whole meaning of instruction for development is based on this, and it in fact constitutes the concept of the "next area of development." Imitation, in a broad sense, is the chief form in which instruction influences development. Learning to talk and learning in school are based to an enormous extent on imitation. For in school the child learns not what he knows how to do on his own, but what he does not know how to do, which becomes comprehensible to him in collaboration with the teacher. *What is fundamental in learning is the very fact that the child is learning something new*. The next area of development, which determines the realm of transitions that are accessible to the child, thus actually turns out to be the most decisive factor in the relationship between instruction and development [9:276] (italics ours -- D. E.).

Vygotskii thus thought that so far as development is concerned, the most effective form of instruction is that which is carried out with the guidance of an adult, the teacher, or in collaboration with the teacher as the bearer of the new material for the child to learn. Zankov, on the other hand, actually limited the role of such instruction in development and shifted the emphasis to instruction in the form of pupils' activity on their own. At first glance it might even seem that Zankov's position is the more progressive and modern one. But only at first glance.

Concerning what might appear to be learning on one's own, Vygotskii wrote:

> After all, when we say that the child is operating by imitation, this does not mean that he looks the other person in the eye and imitates him. If I saw something done today and do the same thing tomorrow, I would be doing it by imitation. When a pupil solves problems at home after he is shown a model in class, he is continuing to collaborate even though the teacher is not standing by him just then. From a psychological standpoint we have a right to consider the solution of the second problem [involving the application of a scientific concept], by analogy with solving problems at home, as a solution arrived at with the aid of the teacher. This aid, this collaboration, is invisibly present and is involved in the child's solution, which he seems to be arriving at on his own [9:284].

We may be further agreed that acquaintance on one's own with an object may result in empirical knowledge of it, and acquaintance with an aggregate of like objects may result in an empirical concept or a general notion. But it is hardly accurate to say that an elementary school pupil can indepedently discover the properties of an object on which the concept of it is based and form a scientific concept on his own. It may be agreed that pupils' activity "on their own" is very important for drilling knowledge which they have already acquired, for exercising it, but not for the actual process of acquiring new concepts, not for the initial discovery of their real meaning.

To limit the types of instruction based on collaboration with the teacher, and to increase the types of instruction based on "independent activity" is in fact to confine elementary instruction to the realm of empirical notions and to reduce developmental processes to exercises. Actually, therefore, fundamental theoretical differences lie hidden behind Zankov's and Vygotskii's differing interpretations of what would seem to be a particular issue about the "next area of development" and its function in instruction.

How, then, is Zankov's statement to be interpreted that "in setting up the learning process in our experimental class we are proceeding from Vygotskii's theoretical views and are structuring this process so that instruction precedes development and thus achieves the optimum results in the development of the intellect" [12:12]. Evidently he accepts only the thesis that instruction should precede development, that instruction plays the key role in development. But he differs radically from Vygotskii in his specific interpretation of the function of instruction.

What is original with Vygotskii is not his general view of the role of instruction in development, but that he saw the source of this role in the content of the knowledge being acquired, in the mastery not of empirical concepts but of scientific ones, which calls for a special form of instruction. Collaboration with the teacher and his guiding role can be seen most distinctly and directly in this form. The teacher is not simply a person with whom the child is collaborating -- he is not the parent, nor is he the kindergarten teacher. The teacher does not simply organize the child's personal empirical experience, nor simply transmit his personal empirical experience. The teacher is the representative of the knowledge accumulated by society. The form instruction takes during the school years is important because it is the form of the child's life in society, the form of the bond between the child and society. This bond should be as clear and distinct as possible.

The essential difference between Vygotskii's and Zankov's views may be located in the way they characterize the relationship between instruction and development. Zankov does not pursue Vygotskii's new conception of instruction and development during the school years. Zankov indicates that the approach to structuring the learning process in which the emphasis shifts to the pupil's independent intellectual activity is more important now than ever before. True, the problem of making instruction more effective has been posed in recent years in numerous studies by practicing teachers, didacticians, methodologists, and psychologists. By more effective instruction is meant both the extent to which the knowledge and skills stipulated in the curriculum are mastered, and the children's mental development. Without going into an analysis of specific studies, it can be asserted that the basic approach to this problem has been to make instruction more active, with an emphasis on children's independent intellectual activity as one of the main features of vitalized instruction methods. This movement has involved great numbers of teachers and has produced positive results in many schools.

However, while the significance of this trend among teachers is being so highly evaluated the reasons for an attempt to vitalize instruction and the possibilities for it should be clarified. Such an evaluation is especially important for the elementary school, where the child's intellectual activity is just beginning a new cycle of development. The reasons, in our opinion, are the following. On the one hand, there is general dissatisfaction with present school instruction while on the other hand, it is impossible to introduce any essential changes into the content stipulated by the curriculum. This situation gives rise to a search for ways to increase the effectiveness of instruction within the existing system of elementary education -- within the content as it is now specified.

This search, significant in itself, is aimed essentially at compensating for shortcomings in the accepted system of instruction and for the limited scope of its content. The more intensively this search is pursued, the sooner the real possibilities of the existing system will become clear. Everything theoretically possible will be "squeezed" out of it, that is, it will be carried to its logical conclusion.

At the same time, it is important to keep in mind that the limited scope of the existing educational system has been determined historically and primarily concerns the content of instruction, which is determined by the goals of education. Therefore even completely "vital methods" and an increased emphasis on pupils' independent intellectual activity cannot change this content. What is more, one should also be aware that negative results can ensue if the effectiveness of instruction is increased and content remains inadequate.

Let us examine some theoretical views spelled out by Zankov which are at the root of his "new system" of elementary instruction. Zankov assumes that this system has already been constructed and tested experimentally, the basic features of which have been described in a number of books and articles. Even so these descriptions lack sufficient data about the actual progress made by the children who are taught according to the new system. This makes it difficult to relate the published material concerning the extent of the children's development with the content of what they were studying, and to ascertain the depth of their learning. What determined the effectiveness of learning and development remains unclear. It is hoped that complete materials will be soon published, for then it will be possible to examine the theory and the system in their interrelations and in operation.

In his statements Zankov touches upon many theoretical issues, and, in particular, on the essence of development. But his statements are very general and are sometimes difficult to correlate with a specific interpretation of the conditions for *mental* development. For example he wrote:

> The correct approach to investigating the development of the child's mind as he learns is closely bound up with an interpretation of development as a kind of unit of opposing tendencies.
>
> Fear on the part of some of our psychologists and educators to resolve firmly to discover the true sources of "self-motivation" results from their apprehension that this might lead to an under-estimation of external, especially educational, influences. There is no basis for such apprehension. Marxist dialectics does not underestimate and certainly does not deny external causes. But external causes operate through internal ones [12:21].

It is unclear which causes of development are considered external ones and which are internal. How is the operation of external causes "through" internal ones to be understood? The following statement by Zankov sheds some light on "self-motivation."

> Our assumption is that during the experimental instruction various types of mental activity are developing -- in particular, analytic observation on the one hand, and isolating and generalizing essential attributes and forming concepts on the other. In each unit of instruction, and in each of the lessons, the type of mental activity alternates. The lessons of one type are separated both in time and by the content of the material being taught.
>
> Each of the particular lines of experimental instruction is having a direct influence as well, of course, in the sense that it is altering one of the types of mental activity. This direct influence is not all that is taking place, however. The internal processes are operating according to their own laws, and the various modes of operation are becoming unified into a functional system. This, evidently, is what determines progress in mental development. . . . The formation of systems involving various modes of operation is evidently the most important line of mental development. . . . In formulating our assumptions, we are relying on Pavlov's ideas about the systematic character of the work of the cerebral hemispheres [12:28].

If it is agreed conditionally that mental development consists basically of the formation of new functional systems (although it is unclear what these are), how then do they come about? It is the task of instruction to develop the particular isolated forms of mental activity (more it apparently cannot do). From this material obtained through instruction, "self-motivation" synthesizes something and produces new functional systems, necessarily of a higher order. The "self-motivation" possessing this magic power is none other than the laws by which the brain functions, constructing functional systems out of the mosaic of separate elements.

If our interpretation is accurate, it is not at all surprising that some psychologists and teachers are wary of such "self-motivation."

To recognize it in this form is naturally to assign instruction a secondary role.

Another interpretation of the facts Zankov cites is possible, however. In actual fact, some lessons cultivate detailed observation, that is, the isolation of as many particular visual attributes of an object as possible, and exercise perceptive activity formed during the preceding stage of development. In other lessons conceptualization is cultivated, that is, a new intellectual activity, thinking, is formed. As a result of the "influence" of thinking, detailed observation is transformed into generalizing observation -- observation mediated by thought. In this possible explanation there is no reference to abstract "self-motivation" nor to laws pertaining to the "systematic nature" of the brain. It is a simple example of the way the previously formed mental processes are rearranged as the thinking process is formed.

The principles of Zankov's new instructional system, which should be the concrete embodiment of his general theoretical views, needs to be examined. But first how it came to be considered necessary for a new instructional system to be set up must be examined. Observation and special investigations testify that primary school pupils' development progresses very slowly. With respect to this fact Zankov said:

> We found this to be so even in classes where the teachers were achieving satisfactory and even good results in imparting knowledge and skills. Real success in teaching knowledge and skills can thus occur unaccompanied by significant changes in the child's development. What results is a scissors effect, a divergence between the knowledge and skills the child has learned on the one hand, and his developmental progress on the other [13:16].

It is difficult to dispute these facts. Zankov continued:

> One may logically conclude that if this is so (as indeed it is), then in order for instruction to stimulate significant progress in the child's development, it is not enough to proceed only from the task of imparting knowledge and skills. The

> fundamentals and methods of instruction must be thought through especially so that both tasks are performed at the same time: attainment of significant progress in the child's development as well as knowledge and skills of a high order of difficulty [13:16].

Zankov expressed the same idea more clearly elsewhere:

> If there is a possibility that a methodological approach which is successful in imparting skills and knowledge may not succeed in terms of the pupil's development, then special direction of the learning process is needed in order for it to be effective for development [11:21].

Agreeing with Vygotskii that instruction must be oriented toward mental functions which have not yet matured, Zankov wrote:

> Although he was correct in emphasizing the role of instruction in forming still undeveloped mental functions in children, Vygotskii did not take into consideration that the pupil's development can vary greatly depending on the way the learning process is set up. For instance, instruction in writing contributes to mental development in varying degrees and effects depending on the method being used to teach it [11:24-25].

The solution which Zankov proposed is similar to Ushinskii's. As a matter of fact, the problem of the divergence between the learning stipulated by the curriculum, and mental development was posed long ago as the problem of "formal education." Ushinskii viewed the problem this way: "Formal development of the faculty of reason in the form in which it used to be understood as taking place is an illusion. Reason develops only through actual knowledge. . ." (quoted in Zankov [11:16]).

But Ushinskii is the very one who worked out special exercises and activities specifically intended for developing logical thinking. There is a contradiction here, of course. He saw that instruction was not providing for sufficient mental development and saw ways of changing it. But he was not able to change the content of academic instruction in any essential way because of historical conditions. He was forced to "compensate" for insufficiencies in the content of what was learned through particular problems and exercises specially aimed at developing the thinking process. But, after all, this was 100 years ago!

Many teachers are following Ushinskii's methods. But is such a division of methods, with particular emphasis on proceeding not "only from the task of imparting knowledge and skills" -- is such a division proper in our conditions, especially in an experimental investigation aimed at creating a new didactic system?

The modern school faces three separate tasks: (1) it must impart a definite volume of knowledge; (2) it must bring about mental development; and (3) it must form cognitive motives. Each of these tasks is performed by its own particular methods. The very fact that performing the first task does not take care of the other two attests to a difficulty that cannot be overcome by still more differentiation in the methods used. But they could be performed all at once, by a single method. The central, determining link in performing all of them is the content of what is being learned, and adequate teaching methods organically connected with it. The solution to this problem provides the basis for solving others -- especially those concerning intensive mental development, the formation of cognitive motives for study, and so forth.

Zankov himself perhaps considers this point as he works out the details of his new instructional system. There are three basic principles in his new system. First, it is necessary to "maintain instruction at a high level of difficulty (at the same time strictly observing the measure of difficulty, of course). Only a teaching procedure which systematically provides abundant material for strenuous mental work can aid the pupils' rapid and intensive development" [11:25]. Second, it is necessary to "go through the instructional material at a fast rate. Thus in each grade not only the curricular material for that particular grade but also what is intended for subsequent grades is studied" [11:25]. Third, "the emphasis on the cognitive side of elementary instruction and on theoretical knowledge needs to be sharply increased" [11:26].[8]

It is unclear whether these principles are given in ascending order of significance, but it is striking that the principle relating to what is being learned (the demand for greater emphasis on theoretical knowledge) comes last.

Zankov indicated that the first principle is closely connected with the second. "This principle [of maintaining instruction at a high level of difficulty] is closely related to a fast rate of progress in the material" [12:40].

The requirement for difficulty may refer to the most diverse aspects of the learning process. For instance, after pupils have learned to add numbers of several digits, they should be given only exercises involving addition of six-digit numbers rather than three- or four-digit ones; or in the study of unstressed vowels, only words which are hardest to check and which need to be examined in their most complex forms should be used; or again, only poetry that is complex in both form and content should be given for memorization, not poetry with simple content. If this is the way the demand for a high level of difficulty is interpreted, then it refers only to exercises. But even when a weight lifter is training, he never exercises with maximum, record loads.

Does difficulty perhaps mean the degree of complexity of the material to be learned? There are grounds for this interpretation as well. "Even if the very best teaching methods are being used," Zankov wrote, "and the pupils are exerting maximum effort and have the necessary preparation, and they still cannot comprehend the material, then it will inevitably be dead weight in their minds" [12:43].

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[8] One further didactic principle is mentioned in the book Pupil Development in the Learning Process: "The necessity for following a sequence of steps as the material is learned is closely bound up with observing the measure of difficulty" [12:44].

The degree of difficulty is thus the degree of possibility for comprehension and has to do with the problem of the content of knowledge, but the sequence of presentation cannot be based only on the degree of difficulty. In general the specification that pupils always be progressing in the material and that they always be finding out something new in the field they are studying is not directly dependent on the difficulty of learning it. Specifying that pupils should always work to the fullest extent of their intellectual powers indicates that development is being interpreted as a function of practice.

The principle of difficulty involves the content of what is being learned, at least to a slight extent, but the specification of a fast rate pertains not to content but to rate as such. In Zankov's opinion, no matter what the material, mental development depends on the rate at which the pupils go through it. For a given unit of time such is probably the case for the simple reason that at a fast rate more material will be covered in that time. But the quality of development itself is hardly going to change in this process. It is a fact that if children go through the present curriculum in three years instead of four, their mental development will keep pace with the increased speed. Whether the quality of mental development changes is problematical and not proven.

If the difficulty of the material and the rate at which the children progress in it have any meaning for development, then it is not direct but only mediated through the content of what is being learned. Thus in essence Zankov, too, is forced to conclude that it is the content that determines mental development. This is reflected in the principle that is last on his list and first in importance. But the way it is formulated elicits doubts. What, in fact, does the stipulation that "emphasis on the cognitive side of elementary instruction and on theoretical knowledge. . . be sharply increased" mean? The "cognitive side" and "theoretical knowledge" are not identical. The "cognitive side" can be broadened significantly and the emphasis on it increased significantly, but the emphasis on theoretical knowledge can be held constant. Analogously, if the amount of empirical knowledge is broadened significantly, theoretical knowledge can be held constant because theoretical knowledge proper is scientific knowledge. The issue is thus actually to increase the emphasis on theoretical, scientific knowledge in elementary instruction. But "increasing" the emphasis on such knowledge is possible only if it is already a part of the content. As there is hardly any such content in the modern curriculum, it needs to be added. In traditional curricula, emphasis on empirical knowledge is all that can be increased.

Curricula are provided in the book Elementary Instruction, but unfortunately they are not detailed so that it is impossible to judge from them what each item covers.

Just to give a few examples, in the curriculum now in effect for the study of Russian in the first grade, words designating objects, actions, and qualities are classified. This classification is based on empirical attributes. In Zankov's curriculum this item is replaced by three others: (1) the noun (term and definitions), (2) the verb (term and definitions), and (3) the adjective (term and definitions). Nothing is essentially changed by giving children the terms and the definitions. Thus if the definition: "Words designating objects are called nouns" is given, the notion of what a noun is remains just as empirical as it was before the term and its definition were introduced. The same may be said of the concepts of "root," "prefix," and "ending."

Zankov wrote that:

> During the first two years of instruction, the children are not given the terms which designate parts of speech ("noun," and so forth), and remain unaware of the formal attributes characteristic of a given part of speech. It comes down to the point that even in the second grade the parts of speech are distinguished in the following manner: "words designating objects," "words designating the actions of objects," and "words designating the qualities of objects." Consequently, even though the pupils are led to generalize (a group of words designating objects, or their actions or attributes), these words are not brought together in the form of a definite linguistic category with its own term and formal attributes. Thus awareness of a relationship among words does not reach a qualitative level.

> Our experience has shown that by the time children are in the first grade, they are already capable of learning a number of terms with good comprehension, and observing the formal attributes of parts of speech [11:76-77].

In our opinion, these data still do not indicate that anything besides an empirical notion of parts of speech is being learned. Neither the introduction of a term nor observation of formal attributes in itself clarifies the nature of the concept being learned.

Zankov criticized modern elementary school curricula for not equipping pupils with the fundamentals of science. "Because they do not," he wrote, "the mental development of pupils is being impeded, since material which does not correspond to the logic of the sciences cannot be learned with comprehension" [11:78]. While this criticism is accurate, there is no proof that this basic deficiency is remedied in the new instructional system. For example, consider the arithmetic curriculum. It has three bases: (1) number and numerical series, (2) awareness of the decimal system and computation methods, and (3) understanding of interrelationships in performing arithmetical operations. In essence these principles are not new as the traditional curriculum contains and accomplishes the same objectives. The empirical observations made on numbers and on the operations with them are simply systematized in a different way in the new system. The concepts of number, of the decimal system, and of arithmetical operations, however, do not change. Of course that is true if one does not think that by singling out terms -- *sum*, *elements*, *difference*, *commutative law*, and so forth -- the content of the corresponding concepts changes and that they are converted from empirical notions to scientific ones. The demand for more emphasis on theoretical knowledge in the content of elementary instruction in the new system is not met. This is quite natural because such a change necessitates special experimental research in educational psychology. Furthermore, without fundamentally changing the actual content of elementary education, it is impossible to solve the basic problem of increasing the effectiveness of instruction by imparting more knowledge and skills thereby intensifying mental development.

The theoretical views which Vygotskii formulated have served as a basis for our experimental research. Of fundamental significance was his idea that instruction fulfills its main role in mental development primarily through *the content of what the child is learning*. The adult -- the teacher -- is the key figure and helps the child to develop ways of operating with objects through which he can discover their essential properties -- those which constitute genuine concepts.

What Vygotskii said about the comprehensive system of instruction which the schools followed until the beginning of the thirties has been already quoted. After the transition to teaching by subjects, the content of what was being learned essentially did not change, even though it was divided into subjects. Instruction still has remained empirical in content. One may say then, with complete accuracy, that instruction consolidates preschool modes of thought and the preschool empirical attitude toward reality. It does not shape a new theoretical attitude nor new modes of thought and does not advance mental development.

Academic subjects proper, in the sense of systems of concepts to be learned in the elementary grades, were not designed. The task of designing them has just now come up and has not yet been satisfactorily resolved.

One of the objections currently being raised to a fundamental change in the content of the elementary stages of instruction is based on *developmental characteristics* of children which limit the possibilities for such a change. But the objectors usually forget that the "characteristics" necessary for learning the new content are themselves *formed in the processes of learning it*. Vygotskii himself noted this: "*The development of the psychological basis for learning fundamental subjects doesnot precede the beginning of instruction but takes place as an integral part of the learning process*" [9:269]. Furthermore, it is usually not taken into consideration that the characteristics of mental activity observed in primary school children at present are themselves a definite result of the existing curricula, which are empirical in content.

At the end of the twenties Vygotskii and his collaborators studied the child's capability for forming concepts on his own. They established three basic stages in concept development: syncretism, complexes, and concepts proper. The stage of complex thought includes five successive forms: (1) the associative complex, (2) the "collection" complex, (3) the chain complex, (4) the diffuse complex, and (5) the pseudoconcept. If we superimpose this on a time scale it turns out that the primary school years are characterized by complex thought with a predominance of so-called pseudoconcepts; concepts proper develop in adolescence.

In their internal makeup pseudoconcepts are typical complex, that is, generalizations in which the child does not go beyond the bounds of visual, immediately perceptible properties of objects. Their external similarity to concepts lies in the fact that some of the attributes on which such a generalization is based could coincide with ones which might be selected and brought together on the basis of an abstract concept.

In the last twenty years Piaget and his collaborators have conducted numerous investigations of the development of the thought process in the child. In these investigations they have established three basic genetic stages of mental development. The first is the preoperational stage, the second is the stage of concrete operations, and the third is the stage of formal operations [6,7]. The early school years are characterized by a predominance of concrete operations, on the basis of which properties discovered through immediate visual experience can be systematized.

Essentially, the stages in the development of concepts indicated by Vygotskii and the stages of mental development established by Piaget coincide, even though they describe mental development from different standpoints -- Vygotskii, from the product standpoint, and Piaget, from the operational standpoint.

The numerous facts corroborating Piaget's data are interpreted by many psychologists and teachers as showing the necessity of these stages for mental development -- necessity in terms of both their sequence and their "distribution" by age level. In our view, these data accurately portray the actual characteristics of the child's mental development. However, they do not make explicit the conditions and causes for this particular "outline" of the formation of the child's intellect. Moreover, they provide a basis for making this outline absolute and turning it into a kind of "constant" found in any course of mental development, no matter what the conditions of instruction.[9] We believe that there is no basis for such an absolute approach. In reality, this outline reflects only the fully defined and specific course of childhood mental development which is taking place in the particular historical forms of a system of instruction (in the broad sense of the word) within which -- at the early stages, in any case -- empirical knowledge is predominant, and modes of learning which are mediated by genuine concepts as elements of the theory of a subject are poorly represented. There is reason to think that a change in the content of instruction and a corresponding change in the "type of teaching," as P. Ya. Gal'perin calls it [4], will influence the "chronological outline" of the development of the child's intellect.

Our position on this issue is that it is a theoretical hypothesis which needs to be tested experimentally to be further substantiated and corroborated (this is the very task we have been pursuing in our specific investigations which in part are spelled out in this book). Moreover, material demonstrating the accuracy and farsightedness of this approach to the problem has already been gathered. Thus, on the basis of special research, Gal'perin has come to the following conclusion:

> When concept formation is taught according to the method of developing corresponding operations by stages, one finds neither complexes, nor

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[9] There is an analysis of Piaget's basic view of the problem of"development and instruction" in the afterword by A. N. Leont'ev and O. K. Tikhomirov to his book about the genesis of logical structures [7:433ff.].

> pseudoconcepts, nor intermediate forms composed of elements of scientific and everyday concepts. The child can neither omit an essential attribute from a concept nor introduce anything nonessential into it. Real concepts are formed successfully and rapidly in the later preschool years, and their range is limited only by the knowledge and skills needed as prerequisites.
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> We believe, therefore that the results of Vygotskii's research retain a dual significance for the present as well. First they show the course of concept formation in the conditions which still prevail today, and second, they describe strikingly the "concepts" children have at the early stages of general development, where this situation probably cannot be altered.
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> At the same time one should not forget, however, that concept formation by stages is revealing incomparably greater potentialities for the later (and perhaps also middle) preschool years than has been realized before, and that the way concepts are formed in contemporary school instruction, which also characterizes a process only by its final result, *should not be considered a standard for mental development or a natural limitation on instruction* [4:22-23] (italics ours).

In our view, these statements can rightfully be applied to Piaget's characterization of mental development as well. At the same time it is necessary to keep in mind that the overall approaches of Vygotskii and Piaget to estimating the child's capabilities for learning new material differ fundamentally. The defenders and followers of Piaget believe that new material can be learned only if it is translated into *the language in which the child himself thinks*, that is, if it is adapted to his *current level* of mental development. Since scientific concepts require operations other than those which have formed in the child's personal empirical experience, the first two years the child is in school are supposedly to be spent teaching him the fundamental operations of logic which underlie the further study of mathematical and other sciences.

It is thus proposed that the child's thinking process be developed first, through special exercises, to the level at which he will be able to master the basic concepts of a particular branch of science, and only then introduce new curricular content (see the book by Bruner [2]). These propositions explicitly are based on the point of view that the development of thought is a process having no direct connection with the content of what is being learned and therefore is independent of instruction.

Vygotskii approached this issue in a fundamentally different way. As already noted, he proposed orienting instruction not toward the aspects of mental development which have already been formed, but toward those which are still forming; not "adapting" the material being taught to existing characteristics of the child's thinking process, but introducing material which would demand of him new and higher forms of thought (within the limits ascertained through specialized research into mental development, of course).

Thus, in order to carry out research into real possibilities for childhood mental development it is necessary, while holding to certain premises, to introduce new material the mastery of which is a very important measurement of these same mental capabilities.[10]

Vygotskii wrote in criticism of Piaget's views:

> For Piaget the indicator of the level of the child's thinking is not what the child knows nor what he is capable of learning, but how he thinks in a field about which he has no knowledge. Instruction and development, knowledge and thought are opposed here in the sharpest way [9:254].

As we see it, the conjunction "and" in the problem of "instruction and development" is neither disjunctive nor contrastive but, on the contrary, copulative. Apart from instruction there is not and cannot be mental development at all. It is the most important, the key condition and source of mental development.

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[10] Some premises underlying new ways of setting up mathematics, Russian language, and manual training as academic subjects are spelled out in subsequent sections of this book. [The latter two are not included in this volume (Ed.).] The materials which describe the way primary school children learn new curricular content serve at the same time as indicators of their intellectual capabilities (not "absolute" ones, of course, but only as correlated with this content and the way it is introduced).

The problem of setting up elementary instruction, that is, expanding the content of its basic subjects, so that it will finally result in the formation of full-fledged concepts, is the subject of special research. In the very processes of determining the psychological premises for setting up elementary instruction material and testing experimentally the possibilities for learning this new material, the potentialities for the mental development of children of early school age are in fact being investigated.

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LOGICAL AND PSYCHOLOGICAL PROBLEMS OF ELEMENTARY MATHEMATICS AS AN ACADEMIC SUBJECT*

V. V. Davydov

Deficiencies in the traditional mathematics curricula for the school are being discussed frequently both here and abroad. These curricula do not embody the basic principles and concepts of modern mathematical science, nor do they provide for the necessary development of childrens' mathematical thought, nor is there continuity from the elementary school through the university.

Studies are being carried out in various countries and by international organizations for the purpose of improving curricula. Proposals are being made for ways of presenting modern mathematical concepts rationally in academic courses (for high school, on the whole). Some of the proposals are unquestionably of great theoretical and practical interest.[1]

A curriculum in its concentrated form conveys the content of an academic subject and methods of developing it in teaching. In essence, therefore, attempts to change a curriculum have to do with a change in the content of the subject and the search for new ways of structuring it. Structuring mathematics as an integrated academic subject is a very complex task, demanding the cooperation of teachers, mathematicians, psychologists, and logicians. Selecting the concepts with which the study of mathematics in school should begin is an important part of solving this general task. These concepts are the foundation on which the whole academic subject is built. Children obtain their general orientation in the reality of mathematics in great part from the initial concepts they learn, which, in turn, has an important influence on subsequent progress in this field. Many of the students' difficulties in mathematics in elementary and high school come about, we believe, first, because what they learn does not correspond to the concepts that actually constitute mathematical structures, and second, because general mathematical concepts are introduced into school courses in the wrong sequence.

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*From Learning Capacity and Age Level: Primary Grades, edited by D. B. El'konin and V. V. Davydov, Moscow, Prosveshchenie, 1966, pp. 54-103. Translated by Anne Bigelow.

[1] See, for instance, the curriculum proposed by V. G. Boltyanskii, N. Ya. Vilenkin, and I. M. Yaglom [4], a survey of American research in this field [40, 50, 47], and others.

Unfortunately, the content of elementary mathematical concepts and the method of introducing them in teaching have not been discussed at any length nor carefully investigated up until now even though this is the only basis on which the curricula now in effect can be thoroughly and critically analyzed, their virtues and major shortcomings pointed out, and new variants in the content of school mathematics projected. Work in this area is further impeded by the fact that curriculum designers as a rule do not take into account, to the degree that they should, modern methods of psychological and logical analysis of the learning process, and they underrate the significance of these methods for structuring mathematics as an academic subject.[2]

In our experimental work on curriculum design [10,12,15], we found it was particularly necessary to determine the most appropriate concepts with which to begin mathematics instruction in school. Ascertainment of these concepts created a number of more general problems. There is the problem of the logical nature of the initial concepts of the discipline itself and their relation to the concepts that are initial in the design of the academic subject. There is the problem of relating scientific defintions to the attributes of the object toward which the person learning these definitions is actually oriented. There is the problem of abstracting the attributes of an object and converting them into concepts, both in the course of elaborating knowledge historically and in the mastery of it by an individual. And there are many other logical and psychological problems.

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[2] Of course, the designers of any curriculum orient themselves toward the psychological and logical aspects one way or another; the question is on which psychological and logical data the curriculum developers are relying, whether they are taking modern methods of psychology and logic into consideration, and how much emphasis they are putting on these data and methods as they structure the academic subject.

Traditional curriculum designers have also had to resolve these issues one way or another. However, curriculum designers prefer to focus not on theoretical and cognitive or logical and psychological matters, but on the mathematical aspect -- problems of connecting the mathematical material itself. As a matter of fact, the discussion of trends in redesigning mathematical education also basically revolves around the amount of mathematical knowledge to be included in (or excluded from) the curriculum (see [48], for instance). Logical and psychological issues again remain in the background, first, because of insufficient exposure, and second, because the opinion prevails that the content of an academic subject -- in spite of its uniqueness -- is a relatively direct projection, simply an undeveloped copy, of certain purely "scientific" information (an original critique of this widely held opinion is given by G. P. Shchedrovitskii [42]).

At the same time, if the strictly mathematical aspect of the curricula is examined, especially the fundamental concepts, much is found that is perplexing from the standpoint of advanced mathematics. The study of mathematics in school begins with natural numbers, the basis for instruction for several years. The selection of this "basis" is usually substantiated by mathematical reasons, by indicating the role of the natural numbers in the system of mathematical knowledge. But the role of the natural numbers in mathematics is not so clear as it at first seems to be. A mathematical analysis was thus called for to bring out some basic features of number as a mathematical concept. It turned out that purely mathematical arguments were less of a factor in "basing" the mathematical curricula on number than were the methodologists' apparently obvious ideas of the "primacy" of certain concepts and the origin and development of abstraction both in the history of knowledge and in the ontogenetic process of the child's mastery of it -- that is, ideas having more to do with logic and psychology than with "pure" mathematics.

Recently, particular attention has been given to basing the school course on set theory when curricula are being modernized (this tendency is quite conspicuous both here and abroad). When this change is made in teaching (particularly in the primary grades, as is observed in American schools, [50]), it will inevitably create a number of difficult questions for educational and child psychology and for didactics, for there is almost no research presently on how the child learns the meaning of the concept of a set (as distinguished from learning counting and numbers, which has been investigated from many angles).

It is worthwhile to examine what is said concerning the concept of a set in mathematical literature, especially because some authors do not acknowledge it as the initial and primary concept. The very basis of mathematics and its initial and general attributes currently are being completely reevaluated (see the studies by N. Bourbaki). This matter is closely involved with defining the nature of mathematical *abstraction* itself and ways of deriving it, that is, with the *logical* aspect of the problem, which must be taken into consideration as the academic subject is being set up.

Material cited below is taken from mathematical sources characterizing the *connection* of the concepts of number and set with other mathematical concepts (the general concept of *structure*, in particular). This is being done not by any means to resolve any mathematical issues as such as most of the issues to be touched upon have already been resolved and made a part of the "general" literature. Rather, it is being done to relate the available solutions to methods of organizing the academic subject, the purpose being to clarify certain *logical and psychological* issues.

Logical and psychological research in recent years (the work of Piaget, in particular) has found a relationship between certain "mechanisms" of the child's thought process and general mathematical concepts. We are making a special study of the characteristics of this relationship and what it means in structuring mathematics as an academic subject (the *theoretical* aspect will be dealt with here, rather than any particular variant of a curriculum).[3] The basic logical and psychological problems which must be discussed before the mathematical course material can be arranged are briefly enumerated at the conclusion of this section.

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[3] Specific problems of organizing the elementary course in school mathematics are dealt with in the next section of this chapter. [See the next article (Ed.).]

The Concept of Number and its Relationship to Other Mathematical Concepts

The natural numbers have been the fundamental concept of mathematics throughout mathematical history. They play a very significant role in all areas of production, technology, and everyday life. Consequently, theoretical mathematicians have set aside a special place for the natural numbers among mathematical concepts. Statements have been made in various ways to the effect that the concept of natural number is the initial stage of mathematical abstraction, and that it is the basis on which most mathematical disciplines are built.[4]

The choice of basic elements for the academic subject of mathematics essentially confirms these general statements. The assumption is made here that as the child becomes familiar with number, he is at the same time discovering the initial features of quantitative relationships. Counting and number are the basis of all subsequent study of mathematics in the school.[5]

There is reason to believe, however, that while these statements justly point out the special and basic significance of number, they still do not adequately convey its relationship to other mathematical concepts, nor do they accurately evaluate the role of number in the process of learning mathematics. Certain significant shortcomings in the present mathematics curricula, teaching methods, and textbooks are primarily the result of this fact. The actual relationship between the concept of number and other concepts especially needs to be examined.

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[4] "Number is the basis of modern mathematics. . ." [8:20]; "All mathematics depends upon the concept of a natural number. . ." [23:12]; "The concept of number is the initial one in structuring the majority of mathematical disciplines. . . . It is thus no accident that the study of mathematics begins with an introduction to number" [22:230].

[5] "The study of whole (natural) numbers is the basis, the foundation, for mathematical knowledge" [43:5]; "Whole (abstract and compound concrete) numbers form the basis of the arithmetic course in elementary school" [38:6]; "In elementary school one first must deal with the concept of (natural) number and the counting operation" [39:6].

For this purpose let us consult E. G. Gonin's book, *Theoretical Arithmetic* [22], which is notable in that a significant portion of it is devoted to setting forth *basic general mathematical concepts* on the basis of which the properties of numerical systems (the subject of theoretical arithmetic) are then brought out.

The initial concepts here, possessing certain properties and relationships, are *set*, *element of a set*, and *subset*. There are certain simple methods of obtaining new sets from those given (*union*, *intersection*, and *difference*). These methods and their properties are designated by a special set of symbols ($A \cup B$ for union; $A \cap B$ for intersection; $A \setminus B$ for difference). The concept of *correspondence* between elements of sets is of great importance. A correspondence between elements of sets A and B determines the mapping of set A to set B, designated, for instance, by the letter *f* (*function* or *unitary operation* also are sometimes spoken of instead of mapping). The special conditions of composition and the identity mapping are introduced (the latter is a *particular case* of a one-to-one correspondence). If a one-to-one correspondence between elements of sets exists, then set A is called *equivalent* to set B. With the introduction of the concepts of *equivalence* and *proper subset of a set*, it becomes possible to define *infinite* and *finite* sets (a set equivalent to some proper subset of itself is called infinite).

Related to the concept of correspondence is the concept of *a relation* determined in a set. Relations possess such basic properties as *reflexivity* (*nonreflexivity*, *antireflexivity*), *symmetry*, *transitivity*, *connectedness*. The concept of *isomorphism* is a generalization of the concept of equivalence of sets. Every set has the property of *power* (equivalent sets have the same power, nonequivalent, differing power). The creation of the system of natural numbers has to do with the necessity for describing this important property of sets.

Along with the relation of equivalence, an important role in mathematics is played by the relation of order (the antisymmetrical and the transitive relations), through which the concept of an ordered set is defined. The continuous and discrete ordered sets are defined by introducing the concepts of section, coterminous element, leap, gap, and others.

The concept of scalar, additive, and additive-scalar value is another very important mathematical concept. The power of a set is a particular case of scalar value.

The concepts of a binary operation and certain properties of it (composition, identity, and associativity), and the inverse operations permit special forms of sets -- groups and subgroups -- to be distinguished. A set with its allied operations of addition and multiplication is in certain conditions a ring. A particular case of a ring is a body (division ring). A special form of a division ring is a field [22:7-96].[6]

Numerical systems are defined on the basis of this chain of concepts. Thus "the discrete well-ordered commutative semiring with a unit element which is not zero is called the system of non-negative whole numbers" [22:97]; "the minimal well-ordered semifield is called the system of non-negative rational numbers" [22:131], and so forth.

If the concepts we have enumerated are examined, several things are noticed. First, the concept of number is related to many concepts which precede it -- the concepts of "set," "function," "equivalence," and "power," in particular. It is only a description of a particular -- if quite important -- property of sets: their power. Thus number is not primary or fundamental in the general structure of modern mathematical concepts. Very important concepts (set, value, group, ring) are introduced before it and independently of it. The properties of numerical systems themselves, in fact, are revealed on the basis of other general mathematical concepts.

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[6] We are simply enumerating the basic general mathematical concepts here in order to show their relation with the concept of number (for definitions of them see Gonin's book [22], or the article by B. Gleigevikht [20]).

Such is the *actual* relation between the concept of number and other mathematical concepts. The reasons for certain categorical statements that the concept of number is primary and that mathematics does not contain the definition of it[7] are therefore not quite clear. If what is meant here is the lack of a satisfactory definition, then this in itself is no reason to assert the "primacy" of number. If the difficulty (or impossibility, even) of defining it within the limits of arithmetic is what is meant, this does not exclude the possibility of a full-fledged definition within the limits of mathematics *as a whole*. If it is assumed that, in its fully developed, finished theory, number is introduced (described) through a system of axioms, this does not mean that broader foundations for the axioms themselves are lacking -- whether in mathematics or in other fields of knowledge (such foundations are being discovered in logic, for instance [23]).

One should keep in mind that the term "definition" has more than one meaning. When it is taken in the sense it has in *formal logic*, the impossibility of setting up such a definition is identified with the "primacy" of the corresponding object, with the impossibility of deducing it. But there are theories of definition in existence now which do not coincide with the traditional approach of formal logic to the matter (see the works of B. M. Kedrov [26], among others).

It should also be mentioned that attempts have been made in the history of science, and a great many attempts are still being made, to provide a *definition* for the concept of number. The definition by Frege-Russell, which stimulated a number of other attempts, is well known (an account of it is given in R. L. Goodstein's book [23]).

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[7] I. N. Shevchenko thus writes: "The concept of number is *primary*. Arithmetic does not provide a definition for it. . . . Mathematics does not contain an answer to the question of what number is, an answer which would consist of a definition of this concept through other, previously established concepts: Mathematics gives this answer in another form, by enumerating the properties of number expressed in axioms" [43:13-14].

Thus the present difficulties, mathematical as well as logical,[8] in defining number are no reason to acknowledge its primacy in the general mathematical system of concepts.

It could be assumed, of course, that even though many preliminary concepts are required simply to describe numerical systems, these systems taken together constitute the subject of mathematics itself in its general features, for something becomes a mathematical phenomenon only insofar as it is expressed in numerical form. But this assumption is not justified. The relation of equivalence, for instance (a reflexive, symmetric and transitive relation), can be found in the equality of segments or in the similarity of figures. Examples of the relation of order (an antisymmetrical and transitive relation) are "smaller" with regard to segments, "younger," for people, and "softer," for minerals [22:27,33]. Here the subject for mathematical consideration is given without being expressed first in numerical form. Seen this way a series of numbers is itself only a special case of these relations.

This state of affairs is not basically at variance with the fundamental significance of the concept of number for mathematics as a whole nor for the study of it. It is important only to correctly evaluate the specific role of this concept and its relationship to other concepts. While an important place is assigned to number in the general system of mathematical knowledge, one should not come to hasty conclusions about the place it should occupy in the mathematics curriculum.

The following situation is typical. Methodologists (N. S. Popova, for example) who think that school mathematics instruction must begin with an introduction to natural numbers themselves still note that the quantitative relations of sets can be taught without having recourse to counting or even being able to name the numbers.

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[8] Goodstein in particular mentions the relationship between the definition of number and logical problems: "The answer to the question 'what is number?' depends at least in part on the answer to the more general question 'what is logic?' We shall see that there are various levels of logic, permitting various definitions of number" [23:12]. (It should be noted that the formally logical "indefinability" of number seems to have to do with only one of these levels.)

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> As we study the ontogenetic and phylogenetic development of numerical ideas, we have become convinced that the concept of number and the counting operation emerge simultaneously when the category of quantity and the category of order interact, although both categories can exist independent of number and counting and independent of each other [39:9].
>
> Even before he can count, the child distinguishes familiar groups of objects in twos and even threes. . . . This direct perception of a set attests to the beginnings of quantitative notions in the child, although at this point he is still a long way from mastering the concept of number [39:11].

These statements acknowledge, on the one hand, that number and counting can be derived from the categories of quantity and order and that the latter are independent of the former. Also, they acknowledge the possibility that the child can conceive of quantity before he masters the concept of number. But again the way the academic subject is set up proceeds from the view that at school "one must first deal with the concept of (natural) number and the counting operation" [39:6]. This approach to the selection of starting points for instruction becomes possible if at least three assumptions are made.

First, it must be assumed that although the categories of quantity and order occur in phylogenesis before number and independently of it, they lose their independence when it appears and are so "displaced" by number that, practically speaking, they cannot be the basis for the formation of mathematical concepts. Number, as the result of the interaction of these categories, embodies them so completely that they themselves can be discovered through numbers, the sequence of which, incidentally, the child learns rapidly and successfully. Their dual nature needs to be distinguished within number and counting [39:14].

Second, before number and counting appear, the quantitative assessment of aggregates in both phylogenesis and ontogenesis bears a prearithmetical character; "prearithmetical operations" have to do with elementary quantitative and ordinal ideas [39:10, 11]. The appearance of arithmetic in phylogenetic development results in conscious counting and full-fledged numerical concepts [39:10]. In ontogenesis, which does not repeat phylogenesis in totality, it is evident one should begin immediately with the formation of "conscious counting" and "full-fledged numerical concepts." The dual nature of numbers and counting requires that the teacher pay special attention to the child's "prearithmetical" training, but in itself, apart from instruction in number and counting, it has no meaning.

Third, the relationsip of number and counting (full-fledged concepts and arithmetical operations) to the categories of quantity and order which occurred prior to them (undeveloped concepts and prearithmetical formulations) permits arithmetic (number) to be made the basis for learning all of mathematics.

In our view these assumptions ignore certain important circumstances, both strictly mathematical ones and logical and psychological ones as well. First, as has been shown above, many general mathematical concepts, concepts of the relations of equivalence and order, in particular, can be dealt with systematically in mathematics independently of numerical form. These concepts do not lose their independent character. With them as a basis it is possible to describe and study a specific topic, that of various numerical systems, whose concepts do not in themselves cover the sense and meaning of the initial definitions. As a matter of fact, in the history of mathematics, general concepts have developed to the extent that "algebraic operations," of which the four operations of arithmetic provide a familiar example, have come to be applied to elements of a totally "non-numerical" character [5:13].[9]

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[9] It is appropriate to cite here the detailed characterization of this process by N. Bourbaki: "The concept of algebraic operation, originally restricted to natural numbers and measurable quantities, gradually broadened parallel to the broadening of the concept of 'number' until it outgrew it and began to be applied to elements of a completely 'non-numerical' character, such as the permutation of a set, for instance. . . . Undoubtedly, the very possibility of these successive expansions, in which the form of the calculations remained constant but the nature of the mathematical objects on which the calculations were being performed changed fundamentally, led to the gradual discovery of the guiding principle of modern mathematics: Mathematical objects in themselves are not so essential -- what are important are their relations" [5:13].

In phylogenesis, people evidently distinguished sets and their powers as objects of certain practical transformations before they did the numerical characteristics proper of aggregates (see, for example, the viewpoint of I. K. Andronov [3:6, 11-12]), but the general concepts of set and power were formulated much later than the attempts to define number theoretically (see the comment by E. G. Gonin [22:13]). Of course, the notion of set and of the relations of equivalence and order did not have the theoretical form in ancient times which modern scientific concepts have. But one should not conclude from this that "prearithmetical" comparisons of aggregates in themselves are less significant than "arithmetical" ones, nor that arithmetical operations are a more "important" form of knowledge than "prearithmetical" description.

This point has to do with difficult theoretical-cognitive and logical problems about the connection among the universal, the particular, and the unique in cognition, and about the relation between practical ("real") and theoretical abstraction. These problems, unfortunately, have not yet been sufficiently worked out in relation to the origin and development of mathematical knowledge. But we can assume that even though arithmetic (numerical systems, laws of calculation, and so forth) was the leading mathematical discipline in a particular period in the development of mankind -- in connection with specific economic needs -- the development of production and of mathematics itself pointed up the limitations of its forms for designating quantitative relationships, and the specific nature of its definitions. For a while it was as though this specific form "outdid" the general features of the subject of mathematics and even appeared to be "loftier". But subsequently these features were expressed in a form specific to them and they revealed a structure which called for special means of description that did not coincide with the arithmetical representation of mathematical relationships. And yet arithmetic itself (the theory of numbers) has come to occupy a new place in the general system of mathematical disciplines; its specific methods and concepts have acquired the necessary relationship to general mathematical and algebraic definitions.[10]

The occurrence of "prearithmetical" behavior in ontogenesis indicated, not a lack of awareness of "quantitative notions," but only a special -- and no less significant -- way of designating and analyzing them which can and should be expanded. And of course, it is necessary to see that the child forms an accurate conception of the relationship between "prearithmetical" and "arithmetical" operations. But attempting to introduce a specific arithmetical form for expressing mathematical relationships as fast as possible distorts the child's conception of these relationships and the relationship between the general and the specific.

There have been recent attempts to extend the stage of instruction at which the child is introduced to mathematics. This trend can be seen in methodological manuals, as well as in some experimental textbooks. For instance, problems and exercises designed to train children to establish the identity of groups of objects are introduced in the first few pages of one American textbook for children six or seven years old [46]. The children are shown how to unite sets, and the appropriate mathematical symbols (the symbols $\cup$ and $+$) are introduced. The study of numbers is based on an elementary knowledge of sets [46:82]. The concrete attempts to move in this direction may not be of equal merit, but the trend itself, we think, is entirely proper and farsighted.[11]

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[10] Let us quote Bourbaki's characterization of the relationship between arithmetic and algebra: "Inasmuch as the set of natural numbers possesses two internal laws of composition -- addition and multiplication -- classical arithmetic (or the theory of numbers), having as its subject the study of natural numbers, is included in algebra. But from the algebraic structure defined by these two laws there emerges a structure defined by the relationship of order 'a divides b'; the very essence of classical arithmetic consists of the study of the relationships between these two structures which occur together" [5:15].

[11] Also among the opinions which have been expressed about improving the mathematics curriculum is a defense of the traditional method of introducing the child to number, a defense of the advisability of starting the school mathematics course directly with number (see the book by N. A. Menchinskaya and M. I. Moro [33:88-89], for instance).

One other circumstance -- the specific nature of mathematical abstraction -- is of great importance in choosing starting points for the school mathematics course. A. N. Kolmogorov praises highly Henri Lebesque's attempt to explain the material content of mathematical concepts, but he criticizes him for underestimating the independent nature of mathematics. In conformity with Engels' views, Kolmogorov emphasized that mathematics

> studies the material world from a particular point of view, that *its immediate subject is the spatial forms and quantitative relationships of the real world*. These forms and relationships themselves, in their pure form, rather than specific material bodies, are the reality which mathematics studies [29:11].

Kolmogorov is speaking here of mathematics as a science, of course, but the matter needs to be taken into consideration in setting up the school subject as well. The curriculum should provide the child with work in which he will be able to "move away" from concrete bodies accurately and at the proper moment, *after having distinguished* their spatial forms and quantitative relations and having given them their "pure form." Only on the basis of this can he develop an accurate understanding of mathematics. But he must develop this "form" through constant relation to specific bodies, operations with which the concepts gain a real material meaning. There is a *contradiction* of a sort here in the elementary stages of mathematics instruction (and not only elmentary, it seems). What the research mathematician has before him in its "pure form" has to be constructed in the child's head. This "form" is not given to him at the start. It must be *derived*, arrived at through a definite course of study.

At the same time it is clear that, for the time being, the child cannot approach the academic material he is beginning to work with from the point of view of "pure" forms and relationships because he does not yet have this point of view. On the contrary, by the time a person has distinguished "pure form," he will be perceiving the material bodies themselves differently.

How can this contradiction be resolved in the course of teaching mathematics? What organization of the course and what method of introducing concepts contributes best to the solution of this problem? Not

The Concept of a Set and its Relationship to Mathematical Structures

Operations on sets and their properties. The concept of a set is introduced in mathematics without being logically defined. What this means is the following. Disciplines primarily have to do with certain objects which are combined into *aggregates*, *classes*, or *sets*. Objects belonging to a set are called *elements* of that set [22:7-8]. Sometimes a set can be described precisely by enumerating all of the elements in it. But for very extensive sets this is difficult or simply impossible to do. The more common method of designating sets consists of citing the rule which lets one determine whether any object does or does not belong to the set. This rule (or requirement placed on objects) is connected with a certain property present only in those objects which satisfy this rule. Consequently, "bound up in each set is a certain property present in those and only those objects which belong to that set" [22:9].[13]

It can be seen by examining this way of introducing "set" that *in itself* there is nothing specifically mathematical about it. Actually, *apart from* a mathematical interpretation of sets, people both in everyday life and in various scientific research are constantly distinguishing classes, aggregates, collection of objects, and separate elements making up these collections. And in each particular case the property according to which the set is distinguished is the essential one. Finding this property ("distinguishing the collection") and relating it to the element (including the element in the set) are the problem for the sciences involved (physics, chemistry, biology, political economy, and others). Rules for designating properties of objects and for distinguishing a certain collection of objects on the basis of these properties were formulated within the bounds of formal logic as far back as ancient times. Every noun, since it is a generalization, designates a certain property and sets apart the class of things even the elmentary units of the course can be set up in a sound way if the answers to these questions are lacking. The greatest defects in traditional methodology come in the way these very issues are solved. It does not reveal, to the extent that it should, the characteristics of quantitative relationships which must be distinguished in order for the initial mathematical abstractions to be structured in the child's mind and for further work to be done on the level of these abstractions.

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[13] "The concept of a *class* or an *aggregate* or a *set* of objects is one of the most fundamental in mathematics. A set is determined by a particular property or attribute. . . which each object being examined either should or should not possess; the objects which possess the property. . . form the set. . ." [8:163].

The issue concerning how to begin the mathematics course and whether it is advisable to begin it directly with number is not a narrow methodological and specific issue, but a fundamental one from the standpoint of developing general notions about mathematics in the child. It may be assumed that the real significance of the elementary stages of instruction in fact consists of showing children the general characteristics of the abstractions constituting the subject of further study and comprising its "pure form." The nature and degree of this "purity" will not directly coincide with the theory of the subject, of course, but there should be some similarity in the content. Determining exactly what the difference and the partial similarity consist of is a subject for logical and psychological as well as educational research.[12] In any case, here is the point from which two paths lead -- either in the direction of real mathematical knowledge or in the direction of its "verbal-symbolic" fictions, which one finds frequently in actual teaching practice.

The material cited above indicates that the general concept of a set occupies a special place in modern mathematics. It is appearing more and more frequently in the literature pertaining purely to the school as well, and is receiving ever greater emphasis as number is introduced. Therefore it is worth our while to discuss the meaning of this concept as one of the possible starting points in the teaching of mathematics.

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[12] The psychological significance which the first stages of the child's acquaintance with linguistic phenomena have for the further study of the Russian lanuage is discussed in Chapter III [of the original book (Ed.)].

corresponding to it (*house*, *person*, and so forth). Simply setting apart an aggregate, a class of real objects, and interpreting them as a "set" is no sign, however, that the approach to objects made in other sciences or in practical activity is specifically mathematical. In mathematics, an important abstraction occurs. For a "set," the nature of the elements *does not matter*; what belongs to the particular set is all that needs to be indicated. But such an abstraction in and of itself is within the bounds of formal logical description and purely logical rules by which certain relations (as in syllogisms, for instance) can be made apart from the "concrete" nature of the objects being examined.[14]

Bourbaki has an interesting idea about the historical role of the concept of a set in modern mathematics.

> We . . . are not touching upon the ticklish issues, semiphilosophical and semimathematical, which have come up in connection with the problem of the "nature" of mathematical "objects." We shall limit ourselves to the comment that the original plualism in our conceptions of these "objects," regarded at first as idealized "abstractions" of sensory experience and preserving all of their heterogeneity, was replaced by a single notion as a result of axiomatic research in the ninteenth and twentieth centuries, by successively reducing all mathematical concepts first to the concept of whole number and then, at the second stage, to the concept of a *set*. The latter, which for a long time was thought to be "original" and "indefinable," was the subject of numerous arguments because of its exceptional generality and the foggy ideas which it elicits in us. The difficulties disappeared only when the concept of a set itself disappeared (and with it all the metaphysical pseudoproblems concerning mathematical "objects") as the result of recent research into logical formalism. From the point of view of this notion of a set, mathematical structures, strictly speaking, become the only mathematical objects [6:251].

There are a number of essential points in this fundamentally important statement. One notices first that *reducing* all mathematical concepts to the concept of a set resulted in difficulties caused by the *exceptional generality and foggy ideas* (this should evidently be interpreted to mean the real properties of objects) which this concept elicits. These difficulties were overcome only with the "disappearance" of the very concept of set. Since "set" is still in very wide use (by Bourbaki among others), this statement apparently refers to the "disappearance" of the original, initial, indefinable character of the concept of a set. *Mathematical structures*, not sets, are the sole mathematical material.[15] The concept of set assumes that these structures have certain properties, even though this might not be at all evident at the start.

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[14] A set is an exceedingly general concept . . . . We conceive of set as something which can be discussed according to the laws of formal logic" [27:79].

R. Courant and G. Robbins have noted a particular circumstance having to do with mathematical research into sets. The mathematical study of sets is based on the fact that sets may be combined by certain operations to form other sets . . . . The study of operations on sets comprises the 'algebra of sets'" [8:108]. These operations are "union" ("logical sum": $A + B$), "intersection" ("logical product": AB) and "complement" (A') of sets [8:110,111].[16]

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[15] The problem of the nature of the subject of mathematics, as well as of the other sciences, has specific philosophical aspects which must be taken into consideration if the whole problem is to be resolved. The most important of these is the recognition of the *objective* existence, *independent* of the position of the investigator, of the very properties, qualities, and relationships of the things being studied which are only reflected in the system of scientific knowledge. On this plane it is quite accurate to ask what mathematics reflects in the real world, and what properties of things it distinguishes, designates, and investigates, that is, what the *nature* of its material is (and this aspect of the matter cannot be dismissed with any arguments about the allegedly "metaphysical" character of the problems occurring here). What is studied by any particular science, including mathematics, is not things in and of themselves, not objects with all their properties and facets intact, but certain aspects, points, relationships, and connections among real things. The structures studied by mathematics might be a particular instance of such relationships.

[16] "The set of all elements each of which belongs to at least one of the sets A and B (designated by $A \cup B$ or $A + B$) is called the union or combination of sets A and B. The set of all elements each of which belongs to A and B ($A \cap B$ or AB) is called the intersection of sets A and B. The set of all elements each of which belongs to A and does not belong to B ($A \setminus B$) is called the difference of sets A and B [22: 12,13,15].

In and of themselves, these three operations are a translation of quite ordinary connections among things into a conventional language. These connections have also been expressed through formal logical structures. In ordinary logical terminology, *union* becomes "either A or B or both" (a particular thing belongs to at least one of the aggregates); *intersection* becomes "both A and B" (this thing belongs to both aggregates); *complement* becomes "not A" (this thing does not belong to this aggregate, which itself is part of another).[17]

In our view the operations enumerated do not, in and of themselves, reveal only mathematical characteristics. The translation mentioned from one "language" to another *cannot*, in itself, reveal a new *quality* of an object. The purely quantitative specificity of objects, that quantitative *relationship* which mathematics investigates in one way or another, does not come to light in this process.[18] Obviously, matters which reflect a specifically mathematical approach to the investigation of sets are actually concealed, left unexpressed at times, in these descriptions of the operations and in the ways they are used in algebra.

This comes to light in the following circumstances. When the operations mentioned are introduced, mathematicians focus *primarily* on the study of their *properties* (or *laws*) which manifest themselves in a *system of equalities*. Courant and Robbins isolate twenty-six of these laws, among which are:

1. $A + B = B + A$,
2. $A + (B + C) = (A + B) + C$,
3. $A + A = A$,
4. $AB = BA$,
5. $A(BC) = (AB)C$,
6. $AA = A$,

and others [8:110].

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[17] The possibility of this translation of formal logical terms into the language of sets is noted by Courant and Robbins [8:112-114] in particular, as well as by Bourbaki [6:12-13].

[18] At the same time this does not eliminate the importance of mathematical symbols for describing logical relationships and the possible "catching" of new aspects of them in this process (the latter are not always clearly indicated, unfortunately).

It should be noted that the first, second, fourth, and fifth laws are externally identical with the commutative and associative laws of ordinary algebra, but the third and sixth laws have no parallels in this algebra.[19]

Thus A. G. Kurosh has written: "The operations of intersection and union of sets are connected by the following mutually reciprocal distributive laws: For any three sets A, B, C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\text{" [28:9].}$$

In the laws cited it can readily be seen that set operations are not isolated one from another but are combined in definite relations. This combination emerges in the form of equalities indicated by a special symbol ("=").

The statement "A is united with united B and C," by itself -- even when extremely abstract elements are assumed -- signifies only the fact of union and says nothing about its properties. But if it is further affirmed that this union is equal to another (that is, to the united A and B, united with C), then a specific property of the operation is revealed, a property designated as the associative law, which indicates that the order in which the sets are united is unimportant for obtaining the final result (the other laws having equality "=" in their formulas can be viewed analogously). But do all real-life aggregates ("sets") come under the associative law (and the other laws)?

The laws of composition and the concept of mathematical structure. Imagine that there are three sets: a pack of old wolves (A), a group of rabbits (B), and a pack of wolf cubs (C), and let them be combined in the following way. First combine B and C. The result of this "union" will be $B \cup C$, for the wolf cubs are hardly going to "devour" the rabbits. Then combine A with $(B \cup C)$. It is quite possible that the old wolves will become occupied with "caring for" the wolf cubs and will not touch the rabbits. The result of the union will be $A \cup (B \cup C)$. But will this be maintained if the order of union is altered, if A and B are combined first and only then combined with C? Obviously, it will not be maintained. The wolves will "devour" the rabbits, and it will turn out that the associative law does not apply:

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[19] "The laws 1 to 26 form the basis of the algebra of sets" [8:112]. "The verification of the laws of the algebra of sets rested on the analysis of the logical meaning of the relation $A \subset B$ and the operations $A + B$, AB, and A'. We can now reverse this process and use the laws 1 to 26 as the basis for an 'algebra of logic.' More precisely, that part of logic which concerns sets or, equivalently, properties or attributes of objects may be reduced to a formal algebraic system based on the laws 1 to 26" [8:112]. (Here the relation $A \subset B$ signifies that A is contained in B, that is, is a subset of B.)

$$A \cup (B \cup C) \neq (A \cup B) \cup C.$$

Only at first glance is this example naive.[20] In fact the introduction of the associative law and other laws obviously assumes a system of limitations on the objects to which they can apply. These limitations can involve the simple "exclusion" of some set of objects from a broader set, or a specific indication of the system of conditions in which the rule being applied "works." But in both instances the process of structuring an abstraction and setting up constructions (of mathematical elements) which can then be the subject of mathematical transformations proper are being dealt with.

From this point of view it is impossible that all real-life ("natural") aggregates of things are inherently mathematical sets, or the inclusion or exclusion, union or intersection of aggregates mathematical operations. Obviously, a real-life aggregate becomes a mathematical set only when it is presented in certain conditions, or under certain "limitations," that is, one must be able to distinguish and abstract certain properties and relations in it (a certain structure, to use Bourbaki's term). What are these properties and how are they isolated in real-life objects and become a subject for mathematical analysis? These questions are of primary significance

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[20] Courant and Robbins have some interesting ideas about the applicability of the laws of arithmetic: "These laws of arithmetic are very simple, and may seem obvious. But they might not be applicable to entities other than integers. If a and b are symbols not for integers but for chemical substances, and if 'addition' is used in a colloquial sense, it is evident that the commutative law will not always hold. For example, if sulphuric acid is added to water, a diluted solution is obtained, while the addition of water to pure sulphuric acid can result in disaster to the experimenter. Similar illustrations will show that in this type of chemical 'arithmetic' the associative and distributive laws of addition may also fail" [8:2].

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when it comes to actually setting up the elementary curriculum.

As a rule, a person who already has a practical knowledge of mathematics is not aware of the way the subject of its operations can be delimited. For this person it is already delimited and has its own particular characteristics.[21] But this subject is still concealed from the child and needs to be distinguished from other aspects of things (physical, chemical, and so forth). Teachers' ideas concerning ways of distinguishing the necessary relations influence the elementary mathematics curriculum and the choice of suitable concepts, means of representing them, and types of exercises.

From a psychological point of view, problems which a person solves by establishing (and mastering) certain ways of operating and distinguishing the necessary attributes and relations among things, are of particular interest. If these problems and ways of operating are known, the instruction process can be organized so that adequate abstractions can be formed soon enough in the child's mind, instead of chains of external verbal designations directly connected with the numerous properties of the things.

Describing these problems and operations is difficult because they have already been removed from the knowledge that has been learned and even seem superfluous. In itself it is as if this knowledge were directly connected with the properties of the things to which it in fact is directly related.[22] To a certain extent this is permissible in dealing with theoretically developed "minds," but unfortunately the type of child who often is confronted by such a problem does not yet know the methods of setting up a particular abstraction. Naturalizing it, making it into an object, thus results in the loss of the ability to "see" the properties of the thing itself on the one hand, and in the limitation of the abstraction itself to the object on the other, that is, the impracticability of the abstraction, no matter how profusely it is illustrated with "concrete" examples.

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[21] Lebesque paid special attention to this matter as it pertains to arithmetical operations: "We assert, for instance, that two and two are four. I pour two liquids into one glass and two into another; then I pour them all into one container. Will it hold the four liquids? 'That isn't fair,' you answer; 'that's not an arithmetical question.' I put first one pair of animals into a cage, then another pair; how many animals will there be in the cage? 'You are being even more glaringly unfair,' you say, 'since the answer depends on the species of the animals. One beast might eat another up; we also need to know whether the count is to be taken immediately or after a year, in which time the animals could die or breed. Essentially we do not know whether these aggregates you are talking about are unchangeable, or whether each object in the aggregate preserves its individuality, or whether there are any objects which disappear and reappear'."

"But what does what you have said mean if not that the possibility for applying arithmetic requires that certain conditions be fulfilled? So far as the rule of recognition . . . that you gave me is concerned, it, of course, is perfect in practice, but it has no theoretical value. Your rule is reduced to the assertion that arithmetic is applicable when it is applicable. This is why it is impossible to prove that two and two are four, which nevertheless is indisputably true, since the application of it has never deceived us" [29:21-22].

This needs to be discussed especially in view of the fact that the term set has been introduced in school courses too superficially and mechanically at times in the modern methods of mathematics teaching. This term is equated with any aggregate of objects as a kind of generic designation (a set of apples, a set of chairs, and so forth), on the assumption that it gives the concept of number a "modern" grounding. In itself, the tendency toward providing this "grounding" is justifiable. But at the same time one cannot simply replace such words as "pile" and "group" with the word "set," deliberately not indicating the system of specific conditions under which real-life aggregates become sets (in particular, the widely used guides for teachers by I. K. Andronov [2,3] suffer from this defect in the way they ground arithmetic in set theory).

Using the concept of a set as the basis for teaching mathematics thus demands a much broader context than the external characteristics of a set which are sometimes described. A set acquires its meaning from and operates within special systems of relationships among particular categories of things. Only through an analysis of these relationships can the set itself be distinguished, that is, the unit possessing these relationships and the laws inherent in them independent of its physical and other "concrete" nature. A set is abstracted as a consequence of isolating certain relationships among arbitrary objects. The laws characterizing these relationships are the "limitations" by which the specifically mathematical features are isolated and abstracted.[23] It is practically a prerequisite for working with the concept of set to be acquainted with these laws.

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[22] In his study G. P. Shchedrovitskii [41] examined in detail the problems of working out methods of expressing "knowledge" in any symbols and the role of certain systems of operations of replacing and correlating the properties of objects with their symbolic analogues, as well as the illusions one thus gets of the immediate nature of "knowledge."

But basing the teaching of mathematics on "the algebra of sets" means devising a totally different academic subject from the one now taught in the schools.[24] It is important to note just how relevant this task is. Inasmuch as attempts are already being made to perform it, and the concepts of "relationship -- structure" are even penetrating psychological theories of the thought process (Piaget), it is advisable to discuss the meaning of these concepts more specifically.

Bourbaki's statement describing the "units" of mathematics as mathematical structures has been cited. But what are structures?

> The common feature of the various concepts having this generic name is that they are applicable to a set of elements the nature of which[25] has not been determined. In order to determine the structure, one or a few relationships by which elements of the set are found are given . . . ; then it is postulated that the given relationship or relationships satisfy certain conditions (which are enumerated and are axioms of the structure being examined) [6:251].

Bourbaki points out three basic types of mathematical structures; algebraic structures, structures of order, and topological structures (while noting that the further development of mathematics may quite possibly lead to an increase in the number of fundamental structures [6:256]).[26] The point of departure for defining a structure consists of relationships, which can be quite diverse.

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[23] Of the two basic components of any 'calculation,' that is, the objects on which operations are being carried out and the rules of the operations, only the latter are really essential. At this higher level of abstraction . . . 'objects' of calculations have a 'nature' which remains almost completely undefined: More precisely, in his calculations an algebra student does not want to know anything about these objects other than the one fact that they obey the laws he is studying" [13:52].

[24] By "basing" we mean the genuine basis of a course -- the logic and content of its foundations and not just those elements of the terminology of set theory and scattered information about sets which are introduced in connection with solving certain methodological problems within the traditional mathematics course.

[25] Bourbaki has a footnote here which we have cited almost in full on page 71.

The ordering principle of modern mathematics as a whole is the *hierarchy of structures*, going from simple to complex, and from general to particular. At the center are the types of structures enumerated above -- the *generative structures*, which are mutually irreducible. Outside this nucleus are the *complex structures*, in which one or a number of generative structures (topological algebra, algebraic topology, theory of integration, etc.) are organically combined. After this come *particular* theories in which the numerous mathematical structures of a more general character collide and interact as at an intersection, the units thereby acquiring "individuality" (the theories of classical mathematics -- analysis, theory of numbers, and so forth) [6:256].

The above, according to Bourbaki, is the architecture of modern mathematics. This architecture is brought to light by moving *from the general*, *the fundamental*, *the productive*, *and the simple to the particular*, *the derivative*, *the complex*, *and the individual*, respectively. The content of complex structures can be correctly understood only through an analysis of the transition within which the original, simple structures are combined organically and interact generating particular and individual ones.

This outline of the development of mathematics as a science has a direct relation to the theories of setting up the academic subject. The characteristics of the elementary, initial structures are of particular significance.

Algebraic structure is defined by the "law of composition," that is, by the relationship among three elements which defines the third element simply as a function of the first two. These laws of composition are of two types, *internal* laws and *external* laws.

"A mapping *f* of a certain subset A of the product E X E into E is called an internal law of composition of the elments of set E. The value $\underline{f}(x,y)$ of mapping $\underline{f}$ when $(x,y) \in A$ is called the image of x and y with regard to this law" [5:17].

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[26] "The distinctive feature of the concepts of modern mathematics, the basis of them, is the primary importance of structures and algebraic operations" [30:56].

The associative and commutative laws have to do with the internal laws of composition. "A certain law of composition $(x, y) \rightarrow x \mathrm{T} y$ of the elements of set E is always called associative if no matter what the elements x, y, and z of E are, $(x \mathrm{T} y) \mathrm{T} z = x \mathrm{T} (y \mathrm{T} z)$" [5:23].

> Let T be a law of composition of the elements of set E. Elements x and y of E are called permutable with regard to law T, if $x \mathrm{T} y$ and $y \mathrm{T} x$ are defined and $x \mathrm{T} y = y \mathrm{T} x$.
>
> "A law of composition T of the elements of set E is called commutative if for any pair (x, y) of elements of E for which $x \mathrm{T} y$ is defined, x and y are permutable" [5:28]. (The symbol T signifies an arbitrary law of composition here.)
>
> "A mapping $\underline{f}$ of a certain set $A \subset \Omega \times E \, B \, E$ is called an external law of composition of the elements of set Ω, called the set of operators (or the area of operators) of the law, and of the elements of set E. The value $\underline{f}(a, x)$, taken as $\underline{f} \, B \, (a, x) \in A$, is called the image of a and x with regard to this law. Elements of Ω are called the operators of the law [5:55].

A full definition of algebraic structure follows.

> Algebraic structure in set E is any structure defined in E by one or several internal laws of composition of elements of E and by one or several external laws of composition of the operators from the sets of operators Ω, Θ, and so forth, with elements of E. These laws are subject to certain conditions (for instance, associativity and commutativity) or are subject to being combined with each other in certain relationships [5:60].

The structure of order is defined by the relation of order.

> This is the relation between two elements, x and y, which is expressed most frequently in the words 'x is less than or equal to y' and which is designated for the general case by $x \, R \, y$. It is no longer assumed here, as it was in the algebraic structures, that this relation simply defines one of the two elements as being a function of the other. The axioms for an order relation are: (a) for all x, $x \, R \, x$; (b) from the relations $x \, R \, y$ and $y \, R \, x$, it follows that $x = y$; and (c) from the relations $x \, R \, y$ and $y \, R \, z$, it follows that $x \, R \, z$ [6:252].

The concepts of "*neighborhood*," "*boundary*," and "*continuity*," to which the idea of space leads, are formulated mathematically in *topological structures* [6:252-253].

Bourbaki's ideas about the "architecture of mathematics" are quite tempting to teachers, logicians, and psychologists. One begins to envision the study of mathematics as being based on general (simple) structures and the academic subject as being developed through the interrelations and interweaving among them. Two aspects of the matter need to be distinguished when discussing the feasibility of this prospect. The first concerns the possibility and advisability of the arrangement of such a course, given the educational goals and instructional methods of the present mass school or the school of the near future. There are standard answers for such an arrangement and standard, usually limited, solutions with which one must agree when the "actual" circumstances are taken into account.

But there is another aspect of the matter as well concerning the exploratory nature of the experimental study of general problems of structuring academic subjects, and mathematics in particular. The ideas inherent in the experimental study of structuring mathematics are of primary significance for they establish the prerequisites for a substantial and justifiable revision of the ideas of traditional education, for working out a new interpretation of the nature of abstraction and generalization, for the connection between general and particular, for ways of developing the child's thought process, and so forth. In other words, research in this field can answer various difficult questions, questions that are important to the present and future school.

A number of foreign publications show that some of Bourbaki's ideas already are being used in one way or another in experimental curricula and textbooks (in certain units of the high school course, mainly).[27] They are reflected to a certain extent, for instance, in the textbook by R. Davis [9] intended for mathematics instruction in the fifth and subsequent grades of the American school (ten- and eleven-year-old children). This textbook is aimed at the study of the elements of axiomatic algebra, the Cartesian system of coordinates (*coordinate geometry*), and functions. The author, in summarizing his own and some other experimental projects, remarked "that fourth-, fifth-, and sixth-graders are more receptive to abstract mathematics and approach the subject with more creativity and originality than do older children" [9:2].

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[27] E. P. Rosenbaum has surveyed some studies showing this trend [40]. An article by A. I. Markushevich [31] contains a critical analysis of similar investigations. Also, see the "memorandum" by a group of American mathematicians [32], as well as [45] and [21].

Some authors believe it possible and advisable to introduce the concepts of finite mathematics, the theory of probability, and the like to children at an early age.[28] The special significance which the general principles of logic have for learning mathematics and other disciplines becomes apparent here. In particular, it has been proposed that the child's first two years in school be especially devoted to introducing him to the operations of logical addition, multiplication, inclusion, and so forth. "These logical operations undoubtedly are the basis of the more specialized operations and concepts of the various branches of science" [7:45]. (We should note, incidentally, that if the child is given this training, the laws of the "algebra of sets" can also be introduced relatively early.).

Essentially, exploratory research in this area can only be *complex*, since it involves mathematical, logical, psychological, and instructional matters. For example, problems concerning the order in which structures are to be introduced, the range of concepts to be studied and the relations among them, the determination of the attributes of these concepts, the differentiation of "general" and "particular" attributes, etc., occur on the *logical and mathematical* levels.

For *psychology* the problem consists particularly of using certain instructional material to reveal the system of the *child's operations* through which he discovers, distinguishes, and learns initial mathematical relationships. In doing this it is important to consider the stages of learning, and the various ways and degrees to which the child learns and uses the concepts.

A third group of issues may be called *psychodidactic*. Could such an experimental curriculum actually be instituted in the school?

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[28] N. Ya. Vilenkin [49] has proposed an original system of exercises which introduce primary school children to the ideas of the theory of probability and ways of working with matrices.

Would it be within the children's capabilities, and at what age (in which grade) should it be introduced? And mainly, what would be the effect on the intensiveness with which mathematics is learned and on the quality of the learning? Psychological knowledge concerning the sources, conditions, and rate of development of the child's thinking will have much to do with answering such questions. These points shall be dealt with, since interesting data have been gathered about them in child psychology.

Psychological Prerequisites for Structuring Mathematics as an Adademic Subject

At first glance it seems as if the concepts of "relation," "structure," "laws of composition," and other concepts having complex mathematical definitions could not be involved with forming mathematical ideas in small children. Of course, the real and abstract sense of these concepts in their entirety and their place in the axiomatic structure of mathematics as a science demand a well-developed, mathematically "trained" mind in order to be learned. But there are concrete psychological data indicating that the child grasps certain properties of things designated by these concepts relatively early.

One should keep in mind that from the moment of birth until the age of seven to ten, the child is developing highly complex systems of general ideas about the world around him and laying the foundation for thinking about objects. In doing so, the child acquires a general orientation toward spatial-temporal and causal-resultant relationships on the basis of relatively limited empirical material. These orientation schemes are a kind of framework for the "system of coordinates" within which the child begins to learn in ever greater detail about the various properties of a multiform world. He is not acutely aware of these general schemes, of course, and cannot express them very well in the form of an abstract statement. To put it figuratively, they are the intuitive form of organization of the child's behavior (although he comes to express them more and more readily in statements, too, of course).[29]

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[29] Many Soviet and foreign authors have studied the preschooler's formation of general orientation schemes. Part of this research has been summarized in the book by D. E. El'konin [14] and in the studies edited by A. V. Zaporozhets and El'konin [52].

In recent decades the Swiss psychologist Piaget and his associates have intensively studied the development of the child's intellect and his general conceptions of reality, time, and space. Some of the studies are directly related to problems of the development of the child's mathematical thinking, and thus it is important that we discuss them as they apply to curriculum design.

In one of his most recent books written in collaboration with B. Inhelder [37], Piaget cites experimental data about the genesis and formation in children (up to the age of twelve or fourteen) of such elementary logical structures as classification and seriation. Classification assumes the performance of the operation of inclusion ($A + A' = B$, for instance) and its inverse ($B - A' = A$). Seriation is the ordering of objects in systematic series (thus, sticks of varying length can be arranged in a series, each member of which is longer than any of the preceding ones and shorter than any of the subsequent ones).

In their analysis of the formation of classification, Piaget and Inhelder show how the child moves initially from the creation of a "visual aggregate" based only on the spatial proximity of the objects, to classification based on the relationship of similarity ("non-visual aggregate"), and then to the most complex form — the inclusion of classes into an hierarchical arrangement as determined by the relations between the extent and the content of a concept. The authors discuss the development of classification not only on the basis of one but on the basis of two or three attributes, and the development of the child's ability to alter the basis of the classification as new elements are added. They also find analogous stages in the formation process of seriation as well.

The specific goal of their investigation was to find regularities in the development of the operative structures of the mind. The property of reversibility (the ability of the mind to move forward and backward) is discussed first. Reversibility occurs when "operations and actions can develop in two directions, and an understanding of one of these directions brings about an understanding of the other, *ipso facto*" [36:15].

According to Piaget, reversibility is the fundamental law of composition inherent in the mind. It has two mutually complementary and irreducible forms: conversion (or inversion, or negation) and reciprocity. Conversion occurs, for instance, when the spatial shifting of an object from A to B can be nullified by moving the object back from B to A, equivalent in the end to the identity transformation.

Reciprocity (or compensation) is the situation in which, for example, after the object is moved from A to B it remains at B but the child himself moves from A to B and recreates the original situation in which the object was next to him. Here the movement of the object is not nullified but is compensated for by the corresponding shift the child himself makes. This is no longer the same form of transformation as conversion [36:16].

Piaget has shown in his studies that these transformations first occur (at the age of ten or twelve months) in the form of sensory-motor schemes. In a series of stages, through the gradual coordination of the sensory-motor schemes and through functional symbolic and linguistic respresentation, conversion and reciprocity become properties of intellectual operations and are synthesized into a single operative structure (from the ages of seven to eleven and from twelve to fifteen). At this point the child can coordinate all the spatial shifts into a single one.

Piaget believes that through psychological investigation of the development of arithmetical and geometrical operations in the child's mind (particularly the logical operations) operative structures of thought can be identified with algebraic structures, structures of order, and topological structures [36:13]. Algebraic structure ("the group") thus corresponds to the operative mechanisms of the mind which come under one of the forms of reversibility -- inversion (or negation). A group has four elementary properties: (a) the composition of two elements of a group also yields an element of a group; (b) one and only one inverse element corresponds to any nonzero element; (c) there exists an identity element; (d) successive compositions are associative. In the language of intellectual operations this means that: (a) the coordination of two systems of operations comprises a new scheme which can be combined with the preceding ones; (b) an operation can develop in two directions; (c) when we return to the point of departure, we find it unchanged; (d) the same point can be arrived at in various ways, but the point itself remains unchanged. Piaget wrote: "In a general sense 'group' is a symbolic translation of certain particular functional properties of thought operations: the possibility of the coordination of operations, and the possibility of recurrence and deviations" [36:16].[30]

A form of reversibility such as reciprocity (transposition of order) corresponds to the structure of order. In the years from seven to eleven and from eleven to fifteen, the system of relations based on the principle of reciprocity results in the formation of the structure of order in the child's mind [36:20].[31]

The facts concerning the development of the child "on his own" (that is, development independent of the direct influence of instruction in school) indicate a discrepancy between historical development of geometry and the stages in the child's formation of geometrical concepts. The latter approximate the order of succession of the basic groups in which topology comes first. According to Piaget's data, topological intuition forms first in the child and then he orients himself toward descriptive and geometric structures. Specifically, as Piaget notes, when the child first attempts to draw, he does not distinguish squares, circles, triangles or other geometric figures, but he does distinguish open and closed figures, a position "outside" or "inside" in relation to a border, and division and neighborhood (without distinguishing distance, for the time being), and so forth, perfectly [36:23].

Since the operative structures of thought form in stages, it is important that we present the scheme of stages that Piaget outlines. From birth until the age of two is the stage of sensory-motor thinking. Conversion and reciprocity already occur in its schemes, but as purely external, motor characteristics of the child's behavior (moving an object away from himself and back, for instance).

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[30] The formation of a logical structure such as classification, involving the inclusion of the part in the whole, presumes algebraic structure, according to Piaget [36:18].

[31] The development of seriation as a logical structure is a process of "discovering" the form of relationship which yields the structure of order.

The stage of visual thinking (the preoperative period), when the child is broadening his knowledge of his environment and is transferring schemes of external operations (with objects) to the level of representation and is becoming capable of performing them mentally (for instance, he begins to carry out mentally the system of transfers which he has done on objects up till now), extends from the ages of two to seven. The mind's capacity for a certain mobility forward and backward improves when objects are being used, although it encounters a number of difficulties.

The stage of concrete operations occurs from the ages of seven or eight to eleven or twelve. The child's mental operations acquire the property of reversibility and a definite structure, but only for solving problems with objects, not the level of "purely" verbal statements. Conversion and reciprocity exist separately. Operations on classes and relationships are still elementary (elementary "groupings").

The stage of formal operations extends from the ages of eleven or twelve until fourteen or fifteen. These operations are performed on the level of "pure" (verbal) statements as well as on problems with objects. The two structures earlier based separately on conversion and reciprocity are synthesized and correspond completely to algebraic structure and the structure of order.

These stages always come in the same order. Piaget sees the source of mental development essentially in the inner logic of the formation of the mind as a particular "system" comparable to an organic system. The real milieu (the social conditions) can hold back or stimulate the course of mental development, but it cannot determine its basic content, direction, or general rate.

Specifically, Piaget believes that mental development is not a direct function of instruction. In fact, opposing tendencies can develop here. For instance, "independent," "spontaneous" development leads the child from topological ideas to descriptive and geometric ones, but the school geometry course begins with metrics. This independent development needs to be taken into consideration, and instruction correlated at the right time with the developing operative structures. Then instruction will accelerate the further development of the child's mind.

Let us examine Piaget's basic views as they pertain to matters of curriculum design.[32] His investigations show, first, that in the preschool and school years, the child is forming operative structures of thought by which he can evaluate the fundamental characteristics of classes of objects and their relationships. Furthermore, at the stage of concrete operations (from the age of seven or eight) the child's intellect is acquiring the property of reversibility, which is exceedingly important for an understanding of the theoretical content of academic subjects and mathematics in particular.

These data indicate that traditional psychology and education have not given enough attention to the complex and capacious nature of the stages of the child's mental development in the periods from two to seven years of age and from seven to eleven years of age.

Piaget himself directly correlates these operative structures with basic mathematical ones. However, although it is completely accurate and justifiable from a factual standpoint to talk about the "translation" of certain properties of a "group" into the language of operations, Piaget has no clear and well founded answer concerning the source of this correspondence. What his position comes down to is essentially that mathematical structures are a formal "continuation" of the operative structures of thought [36:16, 27]. The cause of the correspondence, then, is a genetic relationship between the two types of structures.

This relationship exists because operative structures come about as an abstraction of the operations performed on objects. The content of the abstraction in mathematical logic is of the same nature, as distinguished, for instance, from physical abstraction, which is performed with regard to the properties of the object itself [36:30].

Thus, the source of the "correspondence" between operative and mathematical structures lies in the general type of abstraction (the abstraction of operations). Without going into a discussion of whether such a type of abstraction exists and what its actual characteristics are (there are grounds for assuming that it exists), it is fair to ask the following question. What kind of real objects give rise to these operations which are subsequently abstracted? It is possible to avoid answering this question directly (as Piaget essentially does), in which case the source of the correspondence between the structures is seen only in the particular type of abstraction which they share equally. But if an attempt is made to answer this question, the answer should give an indication of the property the real objects have which, when isolated and "formalized" in an operation, gives rise to both operative and mathematical structures.

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[32] A. N. Leont'ev and O. K. Tikhomirov provide a general analysis of Piaget's conception of childhood mental development, as well as a characterization of his theoretical and cognitive positions, in the afterword to the study mentioned above [37].

Do the operative and mathematical structures have a common "object", and if so, what is it like? Piaget gives no indication of this, because the essence of his view is that no such common object exists. All that the structures of thought and mathematical structures have in common is the type of abstraction. And it is natural that if mathematical structures are a "continuation" of previously formed "operative structures," the child will discover the real subject matter of mathematics only relatively late -- between the ages of twelve and fifteen, when the structures become formal. In other words, mathematical thought is possible only on the basis of formed operative structures (and even then the object of these operations remains in the background).

Thus, the child's development of the operative structures of his mind is not determined by "familiarity" with mathematical objects or by learning ways of operating with them. Rather, the preliminary formation of these structures (as the "coordination of operations") is the basis for mathematical thought and for the "isolation" of mathematical structures.

In the end this solves, to a certain extent, the "tricky" theoretical-cognitive question about the sources of mathematical knowledge. Piaget himself posed it directly: "Does the activity of the mind give rise to mathematical relations, or does it just discover them as a kind of external reality which actually exists?" [36:10]. He does not give a definite answer to this question. On the one hand, he acknowledges the external source of mathematical knowledge, and on the other, after actually comparing operative and mathematical structures, he concludes that "the activity of the mind gives rise to" the latter. A more detailed analysis of his position on this matter is needed; we shall remark only that the answer to this question determines the way in which the sources of mathematical thought, and thus the conditions of its development, are understood.

From our point of view, mathematical relationships are an objective reality, relationships among things that really exist. The activity of the mind just discovers them, and to the degree that it discovers their content, it itself develops. The child appears to encounter these relationships very early. At the age of two or three he is already in fact learning many mathematical relationships of things. These are *spatial-temporal* characteristics of objects having a definite *quantity*. Evidently, as the child becomes familiar with the objects through physical manipulation of them, "operative structures" (in particular, "reversibility") are formed which thereby emerge *from the very beginning* as characteristics of the child's actual *mathematical* thought. This thought is not scientifically mathematical yet, but it does concern mathematical relationships among things. As the child gains further understanding of the relationships among definite quantities of objects, he develops *classification* and *seriation*, which apparently are *practical* transformations of a mathematical nature, that is, not "logical" structures, as Piaget assumes, but practical methods of distinguishing and designating certain mathematical relationships. And "reversibility" is the mechanism for carrying out these methods of operating with objects. In this case it becomes clear why the properties of operative and mathematical structures correspond to each other. The former from the very beginning are formed as mental mechanisms by which the child orients himself to general mathematical relationships.

There is a "genetic relationship" here, too, not based on a common type of abstraction, but based on a *common object*, the orientation to which requires a particular type of abstraction. Of course, genetic (child) psychology confronts a difficult problem -- finding the characteristics of this object, the ways the child "discovers" it, and the reasons he "discovers" the very properties of things which at the height of formal mathematical analysis are described as special relationships and structures. We thus have the experimental problem of determining the causes and conditions of the correspondence, investigated by Piaget in such detail, between the operative structures of thought and mathematical structures.

On the basis of Piaget's results, a number of important conclusions can be drawn about designing the mathematics curriculum. First, the factual data about the development of the child's intellect between the ages of two and eleven indicate that the properties of objects described in the mathematical concepts of "relationship or structure" not only are not "alien" to him at this time but themselves become an organic part of his thinking.

Traditional curricula (particularly in geometry) do not take this into consideration. Thus they do not bring out many of the hidden possibilities in the child's intellectual development. Material from contemporary child psychology supports the general idea of designing an academic subject based on concepts of initial mathematical structures. Of course, there are great difficulties here since there has been no experience yet in designing this kind of subject. One of these difficulties has to do with determining the age-level "threshold" at which instruction according to the new curriculum can begin. If we follow Piaget's logic, apparently these curricula can be used only after children have fully developed the operative structures (at fourteen or fifteen). But if we assume that the child's actual mathematical thought develops within the very process which Piaget designates as the formation of operative structures, then these curricula can be introduced much earlier (at seven or eight, for instance), as the child begins to perform concrete operations with a high level of reversibility. In "natural" conditions, when traditional curricula are being used, it is quite possible that formal operations develop only between the ages of thirteen and fifteen. But cannot their development be "accelerated" through earlier introduction of material which can be learned only by direct analysis of mathematical structures?

We believe that there is such a possibility. The plane of mental operations is already developed sufficiently in children by the age of seven or eight, and through the use of an appropriate curriculum which gives the properties of mathematical structures "openly," as well as the means for analyzing them, children can be brought up more rapidly to the level of "formal" operations than they are when they discover these properties "on their own."

At the same time, there is reason to assume that the characteristics manifested by the thought process at the stage of concrete operations, which Piaget places at the years from seven to eleven, are themselves inseparably connected with the ways instruction is organized in the traditional elementary school. This instruction (both here and abroad) is based on maximally empirical content which frequently has no connection at all with a conceptual (theoretical) approach to the subject.[33] Children's thinking which is geared to the external, directly perceptible attributes of things is supported and drilled through this kind of instruction.

Gal'perin [17,35-36] has noted the connection between the "phenomena" Piaget has discovered in the development of the child's thinking, and the way in which the instruction which develops this thinking in the child is organized. A special investigation by Gal'perin and L. S. Georgiev [18] has brought out an important fact.* They discovered that by changing the organization and content of preschool instruction in elementary mathematical concepts, certain "phenomena" which Piaget had previously found in children of this age consistently disappeared. Of particular significance in their new organization of instruction was the earlier introduction of means of measuring quantities of objects, which "removed" the possibility that children could evaluate quantities of objects only by impression, by the most directly perceptible attribute (the child's primary orientation to directly perceptible attributes is inherent in Piaget's "phenomena," in fact).

In our experimental investigation of the way first-graders count when they have learned about number through the traditional curriculum, we too have discovered a tendency among many of them to evaluate quantities of objects directly. These children orient themselves mainly toward the outwardly perceptible attributes of aggregates of objects, ignoring the foundation in counting they have been given beforehand, which differs from the immediate properties of the elements of aggregates [10]. This situation, which is analogous to Piaget's "phenomena," occurs regularly with the accepted system of introducing the child to number and counting in the schools. But these "phenomena" are removed (simply eliminated) by changing the system, and by reorganizing all of the children's work leading up to the concept of number. If the introduction to number is based from the very beginning on the operation defining the relationship of a whole and a part (any whole and any part), then all children from their first few days in first grade can correctly determine the numerical characteristics of aggregates without "regressing," to estimating numerical characteristics through a direct impression of the aggregates. True, such instruction gives the child another abstraction besides the one he obtains in the traditional curriculum, but this is precisely the task of this different organization. From the very beginning it develops in the child the ability to use special "standards" as means of orienting himself to his surroundings (work having to do with this way of teaching counting is described in detail in this book, as well as in an article by E. S. Orlova [34]).

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[33] El'konin has done an analysis of the empirical nature of the content of elementary instruction [16]. We have indicated some factors explaining the empirical nature of the content of elementary school subjects, in another study [11].

*See Volume I of this series for another research report by Gal'perin and Georgiev (Ed.).

Although Gal'perin notes the great significance of Piaget's investigations, he says:

> In theory the shift from direct thought to mediated thought is the real problem . . . . This shift is dictated not only by the logic of the gradual mastery of "intellectual operations," as Piaget supposes, but by the way the shift to "thinking with tools" is actually organized, by the way the mastery of the use of standards and measures, those real tools of intellectual activity, is organized, and by the way the development of mediated thought, in L. S. Vygotskii's sense, is actually organized [17:36].[34]

In our view, the real characteristics of the child's thought and the stages and rates of its development can be judged only in the context of an experimental solution of this general problem. The rates themselves are derived from the specific ways in which instruction is actually organized. In particular, the rates are derived from the extent to which, and the age levels at which, children master the real "standards" (or conceptual "norms") of mental activity. At the same time, within the context of this problem, answers will be sought to questions about the so-called developmental characteristics of the child's thinking, which in essence can be only relative, dependent on the way the development of thinking is "actually organized" in the instruction process (in the broad sense of the word).

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[34] As far back as the early thirties Vygotskii, the Soviet psychologist, formulated a number of profound theoretical statements about the general conditions for the development of mediated thought and the role of socially elaborated modes of activity ("tools and signs") in this process [51].

The most important aspect of this "organization" is the content of the academic subjects, which in turn is closely associated with the type of instruction (in the sense developed by Gal'perin [19]). By altering the content and the type of instruction in a particular way, the optimal conditions for developing mediated thought can be studied experimentally and the psychological prerequisites for structuring academic subjects thereby brought to light.

Thus facts are now available which show a close relationship between the operative structures of the child's thinking and general mathematical structures, although the "mechanism" of this relationship is far from clear and has been little investigated. The existence of this relationship opens up broad possibilities (for the time being, only possibilities!) for setting up an academic subject which develops "from simple structures to their complex combinations." To make these possibilities a reality it will be necessary to study the shift to mediated thought, and its developmental norms. The method we have mentioned of setting up mathematics as an academic subject may itself be a key factor in developing in children the kind of thought process which has a sufficiently stable conceptual foundation.

Some General Problems of Deciding on the Content of Academic Subjects

From the material cited above we may single out certain key logical and psychological assumptions in the way the traditional mathematics course is set up. First, it is assumed here, in one way or another, that the course must begin with a relatively simple concept, the first abstraction. Number is assumed to be this kind of concept.

We have attempted to show (p. 59) that number is neither simple nor the first in the system of modern general mathematical concepts. Taking it as the "basis" essentially contradicts the assumption itself. The "simplicity" of learning it is not the same as the "simplicity" of the content of the concept of number as the proposed foundation for school mathematics. It is a rather complex abstraction which necessitates many "simpler" bases.[35]

The existence of this contradiction can be seen, for one thing, in the growing tendency at present to introduce other bases into the elementary mathematics course (one of these being the concept of set). One of the arguments in defense of number as the "basis" is to point out that it was first in the history of mathematics itself, as well.[36] This argument reflects another assumption made in setting up the course -- linking the "basis" of it with the history of knowledge. But was this concept "first" in the history of mathematical knowledge? And what is the most advantageous way to approach the history of concepts?

Numerous data indicate that in human history (and in ontogenesis, for that matter; see p. 64), the categories of "quantity," "order," and a number of others appeared and were being used before they were expressed in specifically numerical form. This essential point can hardly be ignored. Further, as mathematical theory shows, certain abstract laws of modern algebra are inseparably connected with the simplest calculations and can be seen in them. "There are few concepts in mathematics which would be more primary than the 'law of composition': It seems to be inseparable from elementary calculations with natural numbers and measurable quantities" [6:64].[37]

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[35] The following statement by J. Dieudonné is striking: "We should note. . . that even though they [the concepts of number, space and time] serve the needs of practice, these concepts are still very abstract. . ." [13:42].

[36] "In the thousands of years of existence of mathematical science mathematical abstraction has gone through three stages . . . . The first stage belongs to the time when mathematical science was conceived -- to the moment of its very beginning. The first basic concept with which mathematics deals -- the concept of number -- was born at this stage of abstraction" [27:11].

The architects of the traditional academic subject ignore "prenumerical," "non-arithmetical" methods of analyzing mathematical relationships. They do not isolate phenomena having to do with composition or the calculations characteristic of them (but not only of them). Both of these approaches are possible only if one already believes that in the history of knowledge itself "whole number" occupies the key position. A definite theory is guiding the study of history here.[38]

But as we have indicated above, there is another theory, in which the concepts of "relation or structure" are of the greatest significance. Through this theory, aspects of the history of knowledge itself which usually are not detected can be brought to light and investigated. In particular, one can trace the close relationship between operations on natural numbers and the laws of composition. That the "calculations" are preserved here is not what is important. What matters is that which is the center of attention of the person performing the analysis -- particular characteristics of particular objects or more general methods of transforming them.[39]

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[37] Bourbaki gives a detailed description of the relationship between the concept of the law of composition and classical mathematical theories, in addition to showing how this concept is gradually set apart and becomes abstract in form [6:64-72].

[38] Bourbaki has investigated the "arithmetization" of mathematics as it relates to a very specific historical level of development of its ideas and means of analysis [6:35-37].

[39] This is how the French educator and mathematician André Lichnerowicz describes what happens if this relationship is ignored:

> It [classical arithmetic] is set forth in early nineteenth-century style and . . . is a kind of amusing worship of operations whose hidden meaning is independent of the numbers with which it is operating. Our pupils, as they come to us, believe in the existence of addition and multiplication which operate in the absolutely infinite universe [30:55].

In empirical history the sequence of the change in calculations was from "number" to "operations." The academic subject is also set up to follow this sequence directly. The thesis -- and it is correct -- about the necessity of beginning the course with the sources of knowledge actually turns out here to mean the subordination of the outline of the academic subject to the external, empirical history of the discipline. This kind of "historicism" turns into external chronologism. In other words, when the problem of the relationship between the historical and the logical aspects of the academic subject is being solved, preference is given to the historical, which frequently is taken in its concrete, empirical form.[40]

This points to still another assumption about the traditional way in which the academic subject is set up. The material in it is arranged so that as the child learns it he gradually forms a generalization, which represents the final sum of progress through the material. In the history of knowledge, general principles (generalizations) emerge relatively late. Thus, it is assumed, the transition to the general notion, to the abstraction, needs to be kept gradual in instruction, too. For instance, the child should first learn the techniques of working with whole numbers (with particular mathematical "objects") and only then shift to working with letter symbols, which reflect more general "objects." For several years the child should store up ideas about particular cases of functional relationships and only later acquire the concept of a function and general ways of describing it.

This arrangement of the academic subject is based on the assumption that the general notion only follows from an aggregate of particular, "concrete" knowledge and crowns it. But in fact, this particular knowledge exists side by side with the general notion and with what

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[40] In different ways, but with equal justification, many authors have criticized this fact. We shall cite some of their statements.

"We need . . . to perfect teaching which right from the beginning will be closer to our science. . . . I do not think that we need to arrange teaching according to an historical scheme in order to achieve this goal" [30:55].

"Many problems which children are solving in the elementary school right now have come down to our day from ancient times. They differ from the problems solved in Babylonian schools only in external form, but not in mathematical content. . . . Excessive interest in arithmetic results in a poor knowledge of mathematics" [49:19].

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came before it.[41] A peculiar situation develops here. In order to learn particular information, there is no need to have a general interpretation of it, but having this interpretation does not change the essence of the particular.

This interpretation of generalization fully corresponds to the way the academic subject unfolds according to the empirical development of scientific knowledge itself. In the real history of science, however, and in the learning process which corresponds to it, the general notion and the abstraction play a different role from the one they are assigned in traditional pedagogy and educational psychology. The appearance of new general ideas in science has an important influence on how its previously original, simple starting points are interpreted. The ideas at the "top" inevitably alter the way of laying the foundation, which itself then reflects the "new" general ideas. The general not only follows from the particular here, but also changes and restructures the whole appearance and arrangement of the particular knowledge which has given rise to it.

As applied to mathematics, this point is expressed in the following statement by Lichnerowicz: "The characteristic feature of mathematics -- to think and rethink everything as a whole -- is the essential difficulty and basic hindrance to teaching by historical outline, but at the same time it is the very guarantee of progress in mathematics . . . . Because mathematics is so general, the original concepts and theorems undergo an inevitable and complete reinterpretation. What appeared during the searching process to be the original stage turns into a simple exercise, from new points of view" [30:55-56].

The academic subject should, of course, correspond to the history of the discipline, but as expressed in theoretical, logical form which, cleared of chance occurrences, concentrates in itself the sources of the discipline as well. Differentiating between genuine historicism and external chronologism in each specific instance is a special research task. We should note that at times it is impossible to go by terminology alone. When Lichnerowicz [30] objects to the dominance of the "historical plane" in teaching, for instance, he essentially has "chronologism" in mind. A defense of the "historical plane," on the other hand, is sometimes a demand for genuine unity of theory and history. For instance, in the preface to the book, The Teaching of Mathematics, it is stated that Dieudonné (a prominent French mathematician) "holds to the idea of introducing mathematical structures according to historical perspective" [13:8]. But if one examines Dieudonné's study itself [13] one makes a remarkable discovery. While he distinguishes definite historical stages of mathematical abstraction, in teaching he categorically opposes blindly following the modes of thought peculiar to the ancients.[42] He calls for a search for the relationship between "historical persepctive" and modern ideas.

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[41] This interpretation of the relation between general and particular, which is found in didactics and specific methodology, has deep roots in the theories of abstraction and generalization, which themselves are rooted in classical sensationism (the study by A. N. Shimina [44] is one which examines the philosophical aspect of this problem).

This is how Dieudonné describes the task of teaching mathematics.

> "We are inclined these days, particularly among teachers . . ., to contrive to conceal or minimize the abstractness of mathematics for as long as possible. This is a big mistake, in my view. Of course, I am not saying that the child should be confronted with very abstract concepts from the very beginning, but that he should learn these concepts according to the development of his mind and that mathematics should appear in its real form when the structures of the child's thought have formed . . . . The essence of mathematical method should become the basis for teaching, while the material being taught should be presented simply as well-chosen illustration" [13:41].

Dieudonné maintains that even though it is important to take the historical perspective of the development of algebra into account, the child should openly be shown the abstract essence of algebra, and should develop a capacity for abstraction and for using its theoretical power.

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[42] In particular, he sharply criticizes the teaching of methods of problem solving by means of reasoning ad hoc each time, methods which were known even to the Babylonians.

"Undoubtedly it is because of the venerability [of these methods] that these rules remain as they are taught in our day, in spite of frequent protestations by mathematicians: If we accept it as a proven fact that at the age of ten, a child cannot understand the mechanism of equations of the first degree with one unknown, then we should wait a few years, but not cram a great number of unnecessary methods into his head" [13:43].

The theoretical expression of the history of knowledge coincides with the gradual discovery of general ideas, with the shift from simple, primary, and "empty" abstractions to complex, derivative, and concrete concepts. Knowledge develops here from the abstract (one-sided, extremely "meager") to the concrete (many-sided, the unity of the diverse). This very path -- the path of ascent from the abstract to the concrete -- corresponds to the theoretical method of mentally reproducing reality, the method developed in dialectic logic.

Here, too, the ways of setting up the academic subject cannot help but have something fundamental in common with scientific thought, since they both have the same goal -- to reproduce concrete knowledge about an object in the person's mind. The school subject has certain features which distinguish it from "pure" science, for its special function is to form the very mental capabilities of individuals for which special didactic methods are necessary. But basically it is similar to theory. Both move from simple to complex, from abstract to concrete, from the one-sided to the unity of the diverse.[43]

The theory of generalization and abstraction thus is closely related to achieving the logical and psychological prerequisites for setting up an academic subject. The choice of initial concepts for an academic subject at a given level of development of the particular science, as well as the principle by which these concepts will be developed, depends greatly on the interpretation of the relationship between general and particular, logical and historical, empirical and theoretical.

The theory of generalization on which the traditional mathematics course is based can be characterized as the process of reducing empirical knowledge to a general, abstract description of it. But one does not find here the reverse influence of abstraction on the "reworking" of empirical, particular knowledge. In essence this theory ignores the special logic which abstraction possesses, the logic of the theoretical form of knowledge by which the concrete can be derived from the abstract and use can be made of the concrete content of the concepts themselves.[44] From this comes the fear of abstraction (see the witty description of this point by Dieudonné [13]), the inability to work with it (the opinion that mathematics is "hard" to learn has become commonplace, after all), and the use of various "tricks" to simplify the teaching of mathematics (its methodology is the most highly developed of all specific methodologies, but even so the traditional course in school has only barely "made it" to the mathematical ideas of the seventeenth century).

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[43] E. V. Il'enkov's book [24], for one, gives a detailed explanation of the dialectical-materialistic theory of the ascent from the abstract to the concrete. In a special study [25] he analyzes the content of certain academic subjects and ways of structuring them from the standpoint of this theory.

The renewal of the search for ways of structuring the mathematics course, and in particular, the investigation of the possibility of setting it up on the basis of the concepts of "relation or structure," presupposes, in our view, another theory of generalization -- a theory which reveals the "mechanisms" of working with the concepts themselves and of working on deriving concrete knowledge through the interrelations among abstractions. Such a theory is the dialectical materialistic theory of the relation among the universal, the particular, and the unique in cognition, and of the forms of theoretical generalization and its relation to the history of cognition. These problems, which were posed by Hegel in his day and subsequently by the classics of Marxism-Leninism, are being analyzed more and more broadly and deeply in our philosophical literature (we refer the reader to the works of B. M. Kedrov [26], E. V. Il'enkov [24], Z. M. Orudzhev [35], and Zh. Abdil'din, A. Kasymzhanov, L. Naumenko, and M. Bakanidze [1], among others).

Psychological and educational research is needed into the way children learn the forms of generalization indicated in this theory, as well as the way to structure academic subjects so as to insure that this very course of generalization is taken. In other words, it is an important task of research to determine the means of developing theoretical thought in children (mediated thought, in psychological terminology), the principle of which consists of the shift from abstract, general definitions to concrete, particular descriptions of an object. This problem needs to be solved in order to set up an academic subject which will satisfy the requirements of modern science. Otherwise, any "revolutions" will result only in superficial changes in the traditional curriculum, which will often contradict its established content, an example of which is the way many methodological studies propose using set theory characteristics. "Set" is a strictly theoretical term having meaning only within a particular system of approaching the mathematical modeling of objects (see p. 58). At present the point of departure for this system is "relation or structure." The problem of finding a means of presenting and explaining this system to children seven or eight years old is really the problem of finding the "beginning" of the mathematics course. But this is precisely what many people avoid, for the introduction of "relations" requires a different logic and a different theory of generalization from the one by which we are usually guided. "Set" (or more accurately, "quasiset") is presented as a direct, external, generic characteristic of aggregates of objects, and thus it is not allowed internal mathematical movement, the chance to "unfold" (incidentally, such "reforms" are readily accepted by the strictest supporters of the traditional mathematics course).

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[44] The processes of "reduction" and "derivation," as they are understood in modern logic, cannot be identified with "induction" and "deduction" in Mill's classical interpretation (see the analysis of these concepts in the studies by Il'enkov [24], G. P. Shchedrovitskii [42] and others).

Any relation (or structure, on a particular level of analysis) is the object of a profound abstraction and at the same time the beginning of a concept (the beginning, and not the end, as the logical traditions of Locke and Mill customarily assert). Special symbolic means are needed to introduce it (the relation) into teaching (see the general characterization of these in Shchedrovitskii's study [41]). A lack of knowledge of special symbolic means seriously hinders the study of the theoretical form of generalization and ways of developing it through instruction. It is important to keep in mind at the same time that relation or structure is learning material of a special type which has not really been studied as it should by education and psychology (Vygotskii noted certain features of it in his day [51]).[45] Structuring mathematics as a modern academic subject and, particularly, determining the actual content of its elementary units, will depend greatly on the performance of complex research into the bases of this type of learning.

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[45] We are speaking here of learning which is taking place in special conditions of purposeful instruction. Piaget has carried out a general psychological analysis of the role of relation or structure in the child's thinking [36, 37] (see the summary of his studies on pp. 83-89).

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THE PSYCHOLOGICAL CHARACTERISTICS OF THE "PRENUMERICAL" PERIOD OF MATHEMATICS INSTRUCTION*

V. V. Davydov

The Fundamental Concepts of School Mathematics and Their Genetic Relationship: A Theoretical Analysis

In the search for the logical and psychological bases for structuring mathematics as a school subject, a distinction was made between two approaches to solving the problem.

On the one hand, the possibility of studying fundamentally new methods of setting up the mathematics course on the basis of the concepts of "relation or structure" was discovered. This approach involves a complex of logical, psychological, and didactic issues whose solution will open the way for a subject radically different from the currently accepted one, both in its content and in its educational goals.[1]

On the other hand, within the framework of the traditional mathematics course there are a number of psychological and logical issues which, if solved, will mean a more rational "companionship" of its basic units and an improvement in the way they are arranged from grade to grade (a change in the relationship between school arithmetic and algebra, for instance). Issues of this type are discussed below.

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*From Learning Capacity and Age Level: Primary Grades, edited by D. B. El'konin and V. V. Davydov, Moscow, Prosveshchenie, 1966, pp. 104-189. Translated by Anne Bigelow.

[1] In the preceding section of this chapter we discussed the bases on which this kind of academic subject might be structured.

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The Origin of Concepts and Its Importance in Structuring the School Subject

The mathematics course (excluding geometry) in our ten-year school is actually broken down into three basic parts: arithmetic (grades 1 to 5), algebra (grades 6 to 8), and elements of analysis (grades 9 to 10). The basis for this subdivision is not clear. Each basic part has its own special "techniques," of course. Those of arithmetic, for instance, have to do with calculations on numbers of several digits, those of algebra have to do with identity transformations and finding logarithms, and those of analysis have to do with differentiation. But are there deeper bases stemming from the conceptual content of each part?

The academician A. N. Kolmogorov has said, "The entire structure of school algebra and all of mathematical analysis can be erected on the concept of real number. . . ." [13:10]. At the same time he made this striking remark:

> The "algebra" being taught in high school, with its approximate extraction of roots, its logarithms, and so forth, is almost closer to a first chapter in analysis (or in an introduction to it) than to pure algebra proper. If modern specialists in algebra succeed in convincing everyone of the necessity of interpreting the word "algebra" in the sense which suits them and which is fully justified logically, but which does not conform at all to school tradition, then we are going to have to raise the question of renaming the subject now being taught in high school as algebra [13:10].[2]

Thus school "algebra" is such in name only. In fact there is no essential difference between the second and third parts of the course (a number of the units of "algebra" in grades 6 to 8 are preparatory to the transition to analysis proper). Of course, the actual relationship between "algebra" and analysis is more complicated and confused, but this is because of the *historical* development of school mathematics as a subject which attempted to satisfy the most diverse and at times contradictory demands.[3]

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[2] Modern ("pure") algebra studies algebraic structures (see the brief description of them in the preceding section of the chapter [the preceding article in this volume (Ed.)]).

This issue is touched upon in a study by Ya. S. Dubnov [6] who, on the one hand, notes that many concepts in "algebra" lead pupils directly toward the basic ideas of analysis (such concepts as *function*, *limit*, and *coordinate*), and on the other hand, laments the lack of an organic relationship between "algebra" and "the new mathematics" (analysis). To emphasize the great need for achieving as complete a relationship here as possible, Dubnov said:

> *The new mathematics should be not an annex built to adjoin the traditional course, but another story put on it, a superstructure for which the foundation of the entire building should be prepared well in advance.* We thereby approach the problem of *preparing* for analysis and analytic geometry. The ideal arrangement of mathematics instruction would make it impossible to determine the point of transition from the old mathematics to the new [6:156].

As we see it, a distinction needs to be made here between two things: the existence in principle of a similarity between "algebra" and analysis, and the degree to which the relationship between analysis and school "algebra," geometry, and trigonometry is *actually* achieved. In traditional curricula the latter has not been developed nearly enough. The former matter, however, is firmly grounded on the concept of *real number*. The following statement by Kolmogorov, commenting on the position taken by Henri Lebesgue is worth noting: "[Lebesgue] is correct in his assertion that from the educational standpoint the school has one indivisible task -- that of providing as clear an understanding as possible of the concept of real number" [13:9-10]. The means of performing this "*indivisible task*" obviously may be varied, but all of them should be in keeping with the final goal facing school "algebra."[4]

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[3] This is how one methods manual evaluates the content of "Algebra": The school algebra course embodies *separate issues* of *various* mathematical sciences: algebra, theoretical arithmetic, and theory of numbers, and *mathematical analysis*. . . . This all goes to show that the school algebra course *does not reflect the unity* which may be seen to characterize the contemporary state of algebra as a science. . . . [14:243] (italics ours -- V. D.).

The next question concerns the basis for distinguishing between school arithmetic and algebra (that is, between the first and second parts of the course). Arithmetic includes the study of natural numbers (positive whole numbers) and fractions (simple fractions and decimals). Special analysis indicates, however, that it is wrong to combine these types of numbers into a *single* academic subject.

The problem is that these numbers have *different* functions. The natural numbers have to do with *counting* objects, while fractions concern the *measurement of quantities*. This is very important to realize in order to understand that fractions (rational numbers) are only *a particular case* of real numbers.[5]

From the standpoint of the measurement of quantities, as Kolomogorov noted:

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[4] The basic difficulty here evidently consists of finding the bases and the form of instruction for "the concept of real number." Dubnov, for one, thought that in contrast to the university course in analysis, which begins with the theory of material (real) numbers, this theory is a "luxury" on the school level because of its difficulty [6:175]. But he probably was questioning only the method of direct exposition of this mathematical theory through lectures, rather than the actual need for designing approaches to the concept of real number.

[5] In theoretical arithmetic the following systems of numbers are distinguished: *complex* (real, or material, numbers plus *imaginary* numbers), *real*, or *material* (rational numbers plus *irrational* numbers), *rational* (whole numbers plus *fractions*), and *whole* numbers. Real numbers (and thus rational and whole numbers) other than 0 are divided into *positive* and *negative* numbers (and besides these there are also systems of *hypercomplex* numbers) [9].

> There is not such a profound distinction between rational and irrational real numbers. For educational reasons children keep studying rational numbers for a long time, because they are easily written as fractions; but the use to which they are put should lead immediately to real numbers in all their generality [13:7].

Kolmogorov thinks Lebesgue's proposal of shifting immediately to the origin and logical nature of real numbers, after studying natural numbers, is justified both on its own merits and from the standpoint of the historical development of mathematics.[6] At the same time, as Kolmogorov noted, "approaching the structuring of rational and real numbers from the standpoint of the measurement of quantities is no less scientific than, say, introducing rational numbers as "ordered pairs," for instance. And for the school it has a definite advantage" [13:9].

Thus there is a definite possibility that "the most general concept of number" (to use Lebesgue's expression), the concept of real number, can be developed directly after a grounding is given in natural (whole) numbers. But what this means in terms of curriculum design is no less than an end to the arithmetic of fractions as it is interpreted in the school. The shift from whole numbers to real numbers is a shift from arithmetic to "algebra," to laying the foundation for analysis.[7]

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[6] This is the way Lebesgue describes his method of introducing numbers: "We went directly from the concept of whole number to *the most general concept of number*, without needing to use or, if you wish, to isolate the concept of an exact decimal or a rational number. . . . In the very same way we shall go directly from an operation on whole numbers to operations on general numbers. . . [13:27] (italics ours -- V. D.).

[7] We are not discussing the relationship between analysis and *theoretical* arithmetic (or its basis), which is defined as the scientific discipline which studies the fundamental properties of *all numerical systems* (or rather, which provides a logical grounding for them).

The meaning of this shift is hidden in actual teaching by the fact that fractions are studied without particular attention to the measurement of quantities. Fractions are given as relationships between pairs of numbers (although the methods manuals nominally acknowledge the importance of the measurement of quantities). As Kolmogorov indicated, introducing fractions in detail on the basis of the measurement of quantities inevitably leads "to real numbers in all their generality" [13:7]. But in fact this does not usually happen, because the pupils are kept working with rational numbers for a long time and are thereby delayed from coming to "algebra."

In other words, algebra in the school begins at the moment when the conditions become right for the shift from whole numbers to real numbers and to the expression of *the results of measurement* in a fraction (a simple one and a finite decimal one and then an infinite one).[8]

Kolmogorov wrote:

> Lebesgue's basic positive educational idea . . . is that mathematical instruction at the various stages of learning can be completely unified. The same concepts, in basically the same form, first are perceived visually through examples, then are formulated more distinctly, and finally are subjected to careful logical analysis.
>
> Infinite decimals are the most suitable approach to this unified exposition, so far as real numbers are concerned. In the elementary school, pupils are introduced to the operation of measurement, they obtain finite decimals from it, and they study arithmetical operations on decimals. The idea that a number may also

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[8] This definition differs substantially from the widely held opinion that "algebra" begins with the introduction of *letters* as symbols (this is expressed distinctly, for instance, in the following statement by V. L. Goncharov: "Arithmetic teaches the use of numbers, while *algebra* teaches the use of letters and formulas" [8:18]). Letter symbols, of course, are of primary significance, but in themselves -- without a change in the *conceptual* foundation -- they cannot be the basis for a new "subject."

> be expressed in an infinite fraction is first broached with the example of periodic fractions which occur in division. In high school, the precision of measurement is discussed in greater detail, the complete correspondence between points on a semistraight line and infinite decimals is established, the general concept of real number is formulated, and the existence of irrational numbers is proven. A logically rigorous exposition along the same general principles is presented in the final year of high school or in the university [13:14-15].

These approaches to introducing real number are interesting from the standpoint of *educational psychology* as much as anything. The "visual" and "manipulative" approach and the "logical" approach to a concept not only are not juxtaposed here, but in fact are genetically connected. The logical approach, essential in the concluding stage of the formation of a concept, becomes evident by the first stage. The genuine unity of mathematics instruction in the school is thus ensured.

The initial steps in unification may include being introduced to the *operation of measuring*, obtaining *finite decimals*, and studying *the operations* on them. If the students have already learned this *form* for recording the results of measuring, it serves as the basis for the idea that a number may also be expressed by an *infinite* fraction.[9] And it is advisable that this basis be established during the *elementary* school years.[10]

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[9] "Of course, practical measurements are always taken only to a finite degree of accuracy, and in order to arrive at a positive confirmation of the irrationality of a relationship . . .it is necessary to go to a higher level of abstraction than the one which corresponds to the naive approximate measurement of quantities. But the possibility of expressing a relationship between two quantities by means of the relationship between two whole numbers is *a chance circumstance, even in the first steps of naive measurement* . . ." [13:7] (italics ours -- V. D.).

[10] In presenting Lebesgue's "curriculum," Kolmogorov has the elementary school of the French educational system in mind, but there are no significant age-level differences between it and our elementary school.

If the concept of fraction (rational number) is removed from school arithmetic, the dividing line between arithmetic and "algebra" will be the difference between whole and real numbers. It is what "chops up" the mathematics course into two parts. We have here not a simple difference but a fundamental "dualism" of sources -- counting and measurement.

By following Lebesgue's ideas regarding the "general concept of number," it is possible to unify mathematics teaching completely, but only after the child has been introduced to both counting and whole (natural) number. Such a preliminary introduction may last varying lengths of time (it is definitely too prolonged in the traditional elementary curricula) and elements of practical measurement may even be brought into the beginning arithmetic course (as is done in the curriculum). But none of this eliminates the difference in the bases of arithmetic and "algebra" as academic subjects. The "dualism" of the points of departure also keeps the sections of the arithmetic course which have to do with the measurement of quantities and the transition to real fractions from really being effective (this apparently is the main reason little attention has been paid to Lebesgue's ideas). Curriculum designers and methodologists are striving to preserve the stability and "purity" of arithmetic as a school subject.[11] The main reason mathematics teaching presents arithmetic (whole number) first, and then "algebra" (real number) is the difference of sources.

This approach seems completely natural and unalterable, and besides, it is justified by many years' experience in the teaching of mathematics. But certain logical and psychological matters make it imperative that this rigid scheme of presentation be analyzed more carefully.

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[11] In a certain sense school arithmetic can be seen as a very simple copy of number theory, the study of natural numbers which remained completely independent even after the broader idea of real number was conceived (see [13:8]).

The fact is that even though these numbers are quite different in form, they are still numbers -- that is, one particular form in which quantitative relationships are represented. First, if it is assumed that the sources and functions of whole numbers and real numbers are absolutely different, how can the fact that both are "numerical in form" be explained? Second, if the real numbers are based on the whole (natural) numbers but their sources are fundamentally different, then how is such a "basis," such a relationship possible? Third, if a special "numerical form" of representation exists, then may it not be assumed that such a "numerical form" has a source of its own which is relatively independent of "particular" forms of numbers and is the forerunner of them?

That whole and real numbers are "numbers" is reason to assume that the very differences between counting and measurement are originally derivative in nature. They have a single source which corresponds to the very form of number. Knowing the characteristics of this single basis of counting and measurement helps one to understand more clearly the conditions in which they originated and the relationship between them.

In order to discover the source of the form of number, one must make a special analysis of the problems man has which he cannot solve without determining the numerical characteristics of some object (through these problems one may establish that numbers are necessary and determine why they are). In another study [2] these problems were examined in detail. It was concluded that, in its general form, number has to do with the need for indirect comparison and assembling of objects. A person may satisfy this need only by first isolating and somehow making a model of the divisibility of the object as a whole by its part (the object may be discontinuous or continuous). When a person is searching for this relationship, he is performing a specific operation. It is the result of it that is represented in the form of a standard aggregate of units (objects and words) which comprise a particular number [2:54-80].

Isolating the "whole" and the "part" depends on the particular "element" of an aggregate (if it is necessary to assemble discontinuous objects). In one problem this element may be the part through which the required relationship is found, while in another problem the very same element will no longer be the basis of the relationship (the basis may now be either a group of separate elements, or a part of the one element, or something else).

But by performing this operation on different objects and substituting one basis (or part) for another, one learns to distinguish the characteristics of these objects and the standard methods of determining their parts. Work on discontinuous objects brings one to a special "technique"-- counting, which is the tool of the study of whole numbers, and, at the same time, produces the concept of whole number. Performing the operation on continuous objects results in measurement and real number.[12]

But developing different "techniques" for performing the same initial operation subsequently conceals this common basis which in turn creates the semblance of a "dualism" of whole and real numbers. If a person learns these concepts as completely finished and theoretically formulated products, he is far removed from their sources, not only from their "distant" ones but even from their "nearest" ones. Such seems to be a standard phenomenon in the formation of concepts and in work with them on the theoretical level.

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[12] Certain writers have noted the connection between the method by which numbers are formed and the establishment of the relationship of the whole and the part; they have indicated two "types" of objects in which this relationship is established; and they have found measurement to be the most representative form in which this operation may be expressed. Descartes wrote, for instance: "The method by which numbers are formed is, properly speaking, a special form of measurement. . . . Considering the parts in their relation to the whole is calculation; on the other hand, considering the whole as divided into parts is measuring it" [4:151].

"The unit of measurement is that universal property (natura) to which . . . all things which are being compared to each other should be applied. . . . There are only two types of things which can be compared to each other: sets and quantities" [4:152-153]. (We should note that by sets Descartes meant discontinuous objects, and by quantities he meant continuous objects.)

Kolmogorov gave a vivid description of this "obliviousness" to the origin of mathematical concepts:

> Mathematicians who already know a finished mathematical theory are inclined to be ashamed of its origins. In contrast to the crystal clarity of the development of the theory, beginning with its already finished basic concepts and assumptions, it seems an unsavory and distasteful job to rummage into their origins [13:10].

With this attitude toward the origin of his own concepts, the theoretician (probably for reasons of his own) is detached from even their closest specific sources and strives to work with "finished," "pure" concepts which, in principle, is possible. "The entire structure of school algebra and all of mathematical analysis," Kolmogorov wrote, "can be erected on the concept of real number without any reference at all to the measurement of specific quantities (length, areas, periods of time, and so forth)" [13:10].[13]

Special "rummaging" is needed just to determine the relationship between real number and the measurement of quantities, to say nothing of more profound relationships. Questions concerning the necessity of determining such relationships and the significance of knowing the origin of concepts to the science itself and to the related school subject arise. However, analysis shows that if concepts are divorced from their sources, in certain conditions, they may lose their content which has an effect on research. Kolmogorov supported Lebesque's view:

> Lebesgue shows how forgetting the actual origin of concepts may lead the researcher astray even in a purely scientific field. . . . Thus the struggle to restore to mathematical concepts their original material content occupies the center of attention throughout his book. To me the basic interest of his book lies in this struggle [13:11].

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[13] The possibility of working this way with a concept has itself developed in the history of science and necessitates certain logical means. N. Bourbaki noted one of the aspects of this: "In his lectures Weierstrass acknowledges the logical interest presented by a complete separation of the concept of real number from the theory of quantities" [1:155]. (Weierstrass was a German mathematician whose studies in the area of real numbers date from the second half of the nineteenth century).

The fact is, "obliviousness to the real origin of concepts" may also be observed in the way the school subject is set up. The authors of textbooks and methods manuals try not to linger over the sources of concepts but to get pupils to work with the concepts themselves as soon as possible, especially as there are opportunities for this.

Kolmogorov wrote:

> A constant trend manifests itself with varying boldness at the various stages of instruction: to get through with the *introduction* to numbers as soon as possible and then to talk only about numbers and the relationships between them. Lebesgue is protesting against this trend [13:10].

What is the reason for this protest, and of what importance is "an introduction to numbers" (and to other matters) in setting up the subject properly? Here is what Kolmogorov has to say: "The problem is not specific defects, but rather that *divorcing* mathematical concepts from their origins, in teaching, results in *a course with a complete absence of principles and with defective logic*" [13:10] (italics ours -- V. D.). Unquestionably he has stated the essence of the matter succinctly here.[14]

It is complicated and difficult to keep the origin of concepts in mind in setting up the entire academic subject. The material content of concepts which acquired their theoretical form long in the past needs special analysis, as do the ways of "transforming" this content into a genuine concept (we need only recall the difficulties which arise in solving these problems with regard to whole numbers and fractions, for instance). The academic subject organized with this requirement in mind will be structured differently from the traditional one, since considerable space in it will be taken up by sections introducing the child to a concept.

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[14] We have purposely quoted extensively in this section from Kolmogorov's preface to Lebesgue's book. We think that his evaluation of Lebesgue's position, as well as his own ideas about such matters as the content of the school mathematics course and the role of analysis of the origin of concepts in setting up the school subject are still of prime importance. Although this preface was first published in 1938 (date of the first edition of Lebesgue's book), his ideas, in our view, have not been used nearly enough either by methodologists or by psychologists (see the study by Dubnov [6:134-135], in which he notes the role of Lebesgue's ideas in contemporary methods of mathematics teaching).

But let us return to the issue we raised earlier about the connection between school arithmetic and algebra. We have advanced the assumption that whole and real numbers have a common root and that their differences are derived from the particular way in which "numerical form" is used to represent the relationship of whole and part. What are the characteristics of this "root," and might the child's introduction to such characteristics be made a special section of the elementary mathematics course, preceding the study of numbers? Attempts shall be made to answer these questions concretely as we go along. Right now it might be noted that we are asking them in order to find a way to introduce numbers so as to ensure that there will be no "Great Wall of China" between whole numbers and fractions (real numbers, that is) later, and that the differences between them will not become absolute. This preliminary section should provide the basis for studying numbers in their organic relationship to each other and should be without that break in time and in mode of introduction that one finds in the traditional courses.

In other words, we are talking about doing away with the "dualism" of whole and real numbers thereby making it possible to minimize the break between arithmetic and "algebra." This in turn will facilitate the genuine unity of mathematics teaching on all levels beginning with the primary grades.[15]

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[15] On p. 114 we cited Kolmogorov's view that introducing the operation of measurement and finite decimals to children even in the primary grades is essential to achieving this unity. We believe that this introduction needs to be comprehensive, starting in the first grade; then certain provisions must be made for a subsequent "natural" transition from whole numbers to fractions.

If the goal of school mathematics is to develop "as clear an understanding as possible of the idea of real number," this goal should be visible in the child's first ventures into mathematics. Rather than be at variance (as is often observed), the foundations of elementary knowledge should be "prestressed" in preparation for the building which will fulfull this goal. We are not talking about starting with the "goal" but about having it dictate the basic development of the entire school mathematics course, from its "ABC's" on. But in order to determine these "ABC's" a special attempt must be made to disclose their material sources, which, as a rule, are not apparent to the people who work with the finished concepts. If Kolmogorov's advice is followed, this is the very time "the distasteful job of rummaging" into the origins of the basic concepts and assumptions should not be shunned.

Where are we to turn, then, to find the common root of the branching tree of numbers? First it is necessary to analyze the concept of quantity. True, "quantity" leads directly to another term -- "measurement." But this does not eliminate the possibility that "quantity" may have a certain meaning on its own. It may be concluded from examining this aspect of the matter that, on the one hand, measurement is related to counting, and on the other, that operating with numbers is related to certain general mathematical relationships and principles. And so, what is "quantity" and of what interest is it in setting up the elementary sections of school mathematics?

The Concept of Quantity and its Place in the School Mathematics Course

In spite of the widespread use of the term "quantity," there is little agreement among mathematicians as to whether it is correct or advisable to use it either for scientific purposes or in teaching. Ya. S. Dubnov has written:

> In contrast to "number," the term "quantity" not only has not become stabilized in teaching, but cannot even be said to have been satisfactorily defined. We are forced to conclude that the term "quantity" is becoming obsolete, just as the term kolichestvo ["number, quantity, amount" - Trans.] began to disappear from mathematical discourse not so long ago (. . . although it has been retained, of course, in general scientific terminology, such as philosophical terminology) [6:141, 142-143].

True, the meaning of the term has not been stabilized, but this in itself is no basis for "nullifying" the term. The question is not the term, of course (actually it could be any term), but the concept behind it. From Dubnov's brief remarks it is difficult to ascertain what is becoming obsolete -- the term, because of its "instability," or the concept, because it is inadequate to the thing (one may conclude from indirect remarks that it is the latter, however). The issue here, apparently, is not only the suitability of the term but a change in the content of the concept which was once designated as "quantity." Certain properties of objects once directly referred to by this term have now become only specific aspects of characteristics which were discovered later but are more fundamental, and are designated by other terms. The old term may lose its meaning, but the properties it formerly specified still remain, losing only their former "place." This is a typical case of the removal of new concepts from their real sources, for as a term "disappears," any properties of objects it may have suggested are sometimes reduced to a minimum. This fact needs to be taken into account as one investigates the origin and material content of mathematical concepts.

All the same one may take issue with the pessimistic view of the fate of the term "quantity," for it is still widely used in theoretical and educational works (see Lebesgue's book [13] or E. G. Gonin's [9], for instance). So far as the concept meant by it is concerned, we believe that there is no basis for shelving it. In fact, mathematicians characterize it as follows:

> Quantity is one of the basic mathematical concepts whose meaning has undergone a number of generalizations in the course of the development of mathematics [12:340].
>
> This . . . theory -- the idea of quantity -- plays a key role in laying the groundwork for all of mathematics [10:109].

Furthermore, the meaning of this concept cannot be said to be "unsatisfactorily defined," either. Kolmogorov has given a clear description of "quantity" [12], and Kagan[16] and other authors[17] have defined and analyzed this concept in detail in their studies. Kagan's approach to the problem of quantity is examined first, since it is the clearest and most consistent.

In common usage the term "quantity" is related to the concepts of "equal to," "more than," and "less than," which are used to describe the most varied qualities (such as length, density, temperature, or whiteness). Kagan wonders what common properties these concepts possess. He shows that they involve aggregates or sets of uniform objects (such as the aggregates of all straight lines, weights, or velocities) to which, when their elements are compared, the terms "more than," "equal to," and "less than" may be applied.

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[16] Kagan spelled out his theory of quantity in an essay written in 1917 (the full version is in an anthology [10]). But he apparently believed that the basic views he had formulated about quantity retained their significance, because he presented this essay in condensed form in one of the chapters of the second part of The Fundamentals of Geometry, which came out in the fifties (after his death, as a matter of fact).

[17] Dubnov notes the existence of axioms for the concept of quantity (see Kolmogorov's article [12]). But in the first place, he believes that it is out of the question to present this complex theory in school (no doubt he means a strictly theoretical presentation), and second, he doubts the very necessity for a generalizing concept of quantity for either geometry or physics [6:142]. He differs on this matter with those who still think that a general concept of quantity is proper and possible. (Unfortunately, Dubnov set forth his interpretation of quantity so briefly that we are prevented from fully understanding the range of his real ideas concerning this problem.)

A set of objects becomes a quantity only when criteria are established by which one may determine, with regard to any elements A and B of the set, whether A is equal to B, more than B, or less than B. One and only one of the relationships:

$$A = B, \qquad A > B, \qquad A < B$$

will hold true for any two elements A and B. These statements constitute a complete disjunction (at least one is true but each excludes all the others).

Kagan distinguished the following eight basic properties of the concepts "equal to," "more than," and "less than."

1. At least one of the relationships $A = B$, $A < B$, or $A > B$ is true.
2. If $A = B$ is true, then $A < B$ will not be true.
3. If $A = B$ is true, then $A > B$ will not be true.
4. If $A = B$ and $B = C$, then $A = C$.
5. If $A > B$ and $B > C$, then $A > C$.
6. If $A < B$ and $B < C$, then $A < C$.
7. Equality is a reversible relationship: $B = A$ always follows from $A = B$.
8. Equality is a reflexive relationship: No matter what element A is of the set under consideration, $A = A$.

The first three statements characterize the disjunction of the basic relationships "=," ">" and "<." Statements 4, 5, and 6 characterize the transitivity for any three elements A, B, and C. The final two statements characterize only equality, its reversibility and its reflexivity. Kagan (following S. O. Shatunovskii) calls these eight basic statements postulates of comparison, on the basis of which a number of other properties of quantity may be deduced.

Kagan described these deduced properties in the form of eight theorems:

> 1. The relationship $A > B$ excludes the relationship $B > A$ ($A < B$ excludes $B < A$).
>
> 2. If $A > B$, then $B < A$ (if $A < B$, then $B > A$).
>
> 3. If $A > B$ is true, then $A < B$ is not true.
>
> 4. If $A_1 = A_2$, $A_2 = A_3$, . . ., $A_{n-1} = A_n$, then $A_1 = A_n$.
>
> 5. If $A_1 > A_2$, $A_2 > A_3$, . . ., $A_{n-1} > A_n$, then $A_1 > A_n$.
>
> 6. If $A_1 < A_2$, $A_2 < A_3$, . . ., $A_{n-1} < A_n$, then $A_1 < A_n$.
>
> 7. If $A = C$ and $B = C$, then $A = B$.
>
> 8. If the equality or inequality $A = B$, or $A > B$, or $A < B$ is true, then it will not be destroyed if we replace one of its elements with an element equal to it (what occurs here is a correlation of the types: if $A = B$ and $A = C$, then $C = B$; or if $A > B$ and $A = C$, then $C > B$, and so forth).

The postulates of comparison and the theorems, Kagan indicated, "cover all the properties of the concepts "equal to," "more than," and "less than" which are applicable in mathematics regardless of the individual properties of the set to whose elements we apply them in various particular cases" [10:95].

The properties indicated in the postulates and theorems may be used to describe many other aspects of objects besides the ones commonly associated with "equal to," "more than," and "less than" (they describe the relationship "ancestor -- descendant," for instance). In describing them a general point of view may be adopted and any three forms of, say, the relationships α, β, and γ may be examined from the standpoint of these postulates and theorems (also, it is possible to determine whether and under what conditions these relationships satisfy the postulates and theorems).

Such a property of things as *hardness*, for instance (*harder*, *softer*, or *identical hardness*), or the sequence of events *in time* (*following*, *preceding*, *simultaneous*) may be examined from this standpoint. In all these instances the relationships α, β, and γ are given a *specific* interpretation. When we choose a set of bodies which will have these relationships, and when we isolate *attributes* by which α, β, and γ may be characterized, criteria of comparison for this set of bodies are being determined (in many instances it is not easy to do in practice). Kagan wrote: "In establishing criteria of comparison, we are converting a set into a quantity" [10:101].

Real objects may be examined from the standpoint of *various* criteria. For instance, a group of people may be studied according to the order in which each of its members was born. Another criterion might be the relative position of their heads when they are standing next to each other on the same horizontal surface. In each of these cases the group will become a quantity with the appropriate designation -- *age*, or *height*. In practice, what is usually designated as quantity is not the set of elements itself but a new concept introduced to distinguish criteria of comparison. This is the origin of such concepts as "volume," "weight," and "electrical tension." Kagan wrote:

> Thus *for the mathematician, a quantity is fully defined when the set of elements and the criteria of comparison have been indicated* [10:107].
>
> *A quantity is thus any set for the elements of which criteria of comparison have been established which satisfy postulates 1 to 8* [10:101].

Kagan views the *natural series of numbers* as a very important example of mathematical quantity. From the standpoint of such a criterion of comparison as the position occupied by numbers in a series (*occupying the same place*, *coming after* . . ., *preceding*), this series satisfies the postulates and is therefore a *quantity*.

According to the same criteria of comparison, an aggregate of fractions also becomes a quantity. Correctly determining the criteria of comparison for the set of irrational numbers (to make it into a quantity) "is the basis of modern analysis" [10:104].

This, according to him, is the content of the *theory of quantity*, which plays such an important part in the basis of all mathematics (we might add that in his essay Kagan proves that the postulates of comparison are *consistent* but *independent*).

Bourbaki regards the *structure of order* as one of the three basic mathematical structures. The *relationship* which determines it, between the two elements *x* and *y*, is given the general designation xRy. But it is most frequently expressed by the words "x is less than or equal to y." The following axioms govern this relatioship: (a) for all x, xRx; (b) from the relationships xRy and yRx, it follows that x = y; (c) from the relationships xRy and yRz, it follows that xRz. The set of *whole* numbers and the set of *real* numbers, for instance, have this structure, "with the symbol "≤" being substituted for R here" [1:252]. Bourbaki remarks especially that one axiom is absent here, the one concerning a property which "seems inseparable from the concept of order we use in everyday life: "whatever x and y are, either xRy or yRx will be true" [1:252].

The three axioms cited above apply to all the forms of the relationship of order, including the case where the elements may turn out to be *incomparable* (where X and Y signify subsets and XRY signifies "X is contained in Y," for instance, or where x and y are natural numbers and xRy signifies "y is divided by x"). But by adding a fourth axiom to them, a special case of the relationship of order is isolated -- the relationship of *comparable* elements so often observed in "everyday life."

The relationship which (according to certain criteria of comparison) Kagan says describes quantity is a *particular instance* of the structure of order. Only the relationships designated by the symbols "=," ">," and "<" figure in Kagan's postulates; nothing is mentioned about any operations with the elements involved. In contrast, the axioms Kolmogorov gives [12] contain the properties of *addition and subtraction* as well as the properties of comparability.

First, Kolmogorov says that as mathematics has developed, the meaning of the concept of quantity has undergone a number of generalizations. The properties of quantities now called *positive scalar quantities*, to distinguish them from subsequent generalizations, were described even in Euclid's *Elements*. Kolmogorov gives the axioms for these quantities and notes that the original conception of them was actually a direct generalization of more concrete notions: length, area, volume, weight, and so forth. Each specific type of quantity is related to a particular *method of comparing* physical bodies or other objects (in geometry, for instance, segments are compared by means of superposition).

A relationship of inequality is established in the system of all uniform quantities. In the case of lengths, areas, volumes, and weights, how the meaning of the *operation of addition* is established is known. The relationship $a < b$ and the operation $a + b = c$ possess the following properties:

1. Whatever a and b are, one and only one of the three relationships $a = b$, $a < b$, or $b < a$ holds true.

2. If $a < b$ and $b < c$, then $a < c$ (the transitivity of the relationship).

3. For any two quantities a and b there exists a single particular quantity c to which $a + b$ is equal.

4. $a + b = b + a$ (the commutativity of addition).

5. $a + (b + c) = (a + b) + c$ (the associativity of addition).

6. $a + b > a$ (the monotony of addition).

7. If $a > b$, then there exists one and only one quantity c for which $b + c = a$ (the possibility of subtraction).

8. Whatever the quantity a and the natural number n are, there exists a quantity b such that $nb = a$ (the possibility of division).

9. Whatever the quantities a and b are, there exists a natural number n such that $a < nb$.[18]

10. If the sequence of quantities

$$a_1 < a_2 < a_3 < \ldots < \ldots < b_3 < b_2 < b_1$$

possesses the property that for any quantity c with a large enough number n,

$$b_n - a_n < c,$$

then there exists one single quantity x which is larger than all a_n and smaller than all b_n (the property of continuity).

Kolmogorov writes: "Properties 1-10 defined the totally modern concept of the system of positive scalar Q [quantities]. If we choose some quantity ℓ as the unit of measurement in this system, then all other Q of the system are represented identically by

$$Q = \alpha\ell,$$

where α is a positive real number" [12:340].

The system of all real numbers possesses all the properties of scalar quantities, so therefore "it is quite right to call the real numbers themselves quantities. It is particularly appropriate when variable quantities are being discussed. . . . This is a logical point of view: Numbers are particular cases of Q just as lengths, volumes and so forth are, and like all Q can be both variable and constant" [12:341].

If Kagan's postulates are compared with the first and second properties of Kolmogorov's axioms, we can see that what the axioms express in brief form (first and foremost, the complete disjunction and the transitivity of relationships) in essence is systematically expanded in the postulates. But in addition, the axioms include a number of other very important properties of quantities pertaining to the possibility and identity of addition, to its commutativity, associativity and monotony, and also to the possibility of subtraction (properties 3 to 7).

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[18] The theory of the measurement of quantities developed by the ancient Greek mathematicians is based on the ninth property along with the more elementary properties 1-8 (see [12:340]).

It is worth noting that what these properties are describing is quantities (positive scalar ones), which may be discussed apart from and before being expressed in numbers. That is, if these properties are kept in mind one can work with real lengths, volumes, weights, periods of time, and so forth (having first established these parameters on objects according to criteria of comparison, of course).

In working with quantities (it is advisable to designate their particular values by letters), a complex system of transformations can be produced through which the relations among properties of the quantities can be determined. In producing the transformations, one may move from equality to inequality and perform addition (and subtraction) -- with the commutative and associative properties as a guide in the addition. For instance, if the relationship $A = B$ is given, then knowing $B = A$ can be of help in "solving" problems. As another example, given the relationships $A > B$ and $B = C$, one can conclude that $A > C$. Or again, since for $A > B$ there exists some C such that $A = B + C$, it is possible to find the difference between A and B ($A - B = C$). All these transformations can be performed on physical bodies and other objects when one has established criteria of comparison and the correspondence between the particular relationships and the postulates of comparison.

The following point deserves special attention. Properties 3 to 6 of the operation of addition describe what Bourbaki defines as algebraic structure.[19] Actually, the relationship which yields

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[19] See the description of this and the other structures on pp. 79-80.

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this structure is a function. That is, it is a relation between two elements such that a third is determined (law of composition [1]). If for any quantities a and b there exists a third c (not necessarily different from a or b) to which a + b is equal, then this is a very simple case of composition, which possesses two inner laws -- commutative and associative (according to the operation of addition. Furthermore, with the introduction of the eighth property, which pertains to multiplication, it becomes possible to apply composition to that operation as well). The axioms cited above thus describe quantity both according to the relationship of order and according to the relationship designated as function (or composition). These are important general mathematical relationships [1:252].

Let us examine properties 8 to 10. The concept of natural number does not explicitly appear before the eighth axiom, which establishes the possibility of division. For any quantity a and natural number n there exists a quantity b such that $nb = a$. This formula can be transformed so that division goes from possible to actual: $n = \frac{a}{b}$, where n is a natural number. If the abstract meaning of this formula is compared with the actual process of finding the relationship between quantities a and b, one may conclude that a natural number can be obtained not only by "counting" but also by "dividing" the quantities, which in fact is the simplest way of measuring them.[20]

The latter fact is of special importance because it rules out the excessive contrasting of quantity to natural (whole) number. There is a more profound connection between quantity and natural number -- through a chain of intermediate links -- than is customarily assumed in traditional methods of teaching. Particular attention needs to be given to the bases of the connections between counting and measurement and between natural (whole) number and the properties of quantities. The connection between the properties of discontinuous sets and those objects which turn into a quantity under certain conditions also becomes particularly important.

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[20] Of interest in this connection is the following statement from M. E. Drabkina's book about the foundations of arithmetic: "The notion of the first natural numbers appeared at the earliest stages of human development in connection with counting the objects in some aggregate and has to do with the measurement of quantities which contain the unit of measurement a whole number of times" [5:5] (italics ours -- V. D.).

Real number is based on positive scalar quantities, the concept of which is defined by all ten of the properties. Some of the properties are essential to natural numbers as well. It is striking that natural numbers, fractions (rational numbers), and real numbers themselves can be represented as quantities (both Kagan and Kolmogorov mention this).

It may be concluded from the material cited above that natural and real numbers are equally closely related to quantities and certain of their essential characteristics (properties 1 to 7). Might not the child study these and the other properties as a special topic before he is introduced to the numerical form for describing the relationship between quantities? These properties could be the basis for a subsequent detailed introduction to number and its various forms (fractions in particular) and such concepts as coordinates and function, even in the early grades.

This introductory section could consist of an introduction to physical objects and criteria for comparing them, with quantity being distinguished as a subject for mathematical consideration. Further, it could be an introduction to methods of comparison, symbolic means for designating the results, and methods of analyzing the general properties of quantities. This section needs to be expanded into a relatively detailed curriculum and, most important, this curriculum needs to be related to actions the child can perform in order to learn the material (in a suitable form, of course). At the same time we need to determine experimentally whether seven-year-old children are capable of mastering this curriculum and whether it is advisable to introduce it, from the standpoint of attempting to bring arithmetic and elementary algebra together in subsequent mathematics teaching in the primary grades.

The Experimental Introduction of the Concept of Quantity in the First Grade

The Content of the Experimental Curriculum

Up to now the discussion has been theoretical and has involved clarifying the mathematical basis for an elementary section of a course designed to introduce the child to the basic properties of quantities (before number is specifically introduced). However, for several years instruction has actually been organized according to such a curriculum for this section and used in our research in experimental classes. So the curriculum described below has been influenced by the results of experimental instruction by one or another of its preliminary variants.

The basic properties of quantities were described earlier. It is senseless, of course, to give seven-year-old children "lectures" about these properties. A way had to be found for the children to work with the instructional material so that they could, first, discover these properties in the things around them, and then learn to designate the properties using certain symbols and to carry out an elementary mathematical analysis of the relationships they had found.

Thus the curriculum should contain first, an indication of the properties of the subject which are to be learned; second, a description of the teaching materials; and third -- the most important psychologically -- a description of the operations by means of which the child distinguishes the particular properties of the subject and learns them. These "components" make up the curriculum in the true sense of the word.[21]

It seems reasonable to spell out the specific features of our curriculum and its "components" by describing the actual instruction process and its results. An outline of the curriculum and its main topics are given with a brief statement concerning the basis for each topic and an explanation of it.

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[21] It should be noted that curricula are usually reduced to a list of topics, with everything else being designated as methods. This division seems wrong to us, at least for new material which is only being experimentally tested.

Topic I -- Comparing and assembling objects (according to length, volume, weight, composition, and other parameters).

1. Practical problems of comparing and assembling.
2. Isolating attributes (criteria) by which the same objects may be compared or assembled.
3. Verbal designation of these attributes ("by length," "by weight," and so forth).

These problems are solved on instructional material (such as boards or weights) by choosing a "similar" object, and reproducing (constructing) a "similar" object according to the parameter designated.

Topic II -- Comparing objects and designating the results in a formula of equality or inequality.

1. Problems of comparing objects and designating the results symbolically.[22]
2. Verbal designation of the results of a comparison (the terms "more than," "less than," and "equal to"). The written symbols ">," "<," and "=."
3. Making a drawing to designate the results of a comparison (first a "copy," and then an "abstraction" -- using lines).
4. Using letters to designate the objects being compared. Writing down the results of a comparison using the formulas: $A = B$, $A < B$, $A > B$.[23]

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[22] We discuss this problem specifically (as distinguished from practical problems of comparing and assembling, for instance) elsewhere [2:67-68].

[23] Various letters of the Russian alphabet (printed capital letters) were used in the formulas. The children were introduced to the Latin alphabet during the second semester.

5. The impossibility of using different formulas to designate the results of a comparison. The choice of a particular formula for a *given* result (the complete disjunction of the relationships *more than*, *less than*, and *equal to*).

Topic III -- The properties of equality and inequality.

1. *The reversibility* and *reflexivity* of equality (if $A = B$, then $B = A$; $A = A$).[24]

2. *The connection between the relationships* "more than" and "less than" in inequalities when the sides being compared are "transposed" (if $A > B$, then $B < A$, and so forth).

3. *Transitivity* as a property of equality and inequality: if $A = B$, and $B = C$, then $A = C$; if $A > B$, and $B > C$, then $A > C$; if $A < B$, and $B < C$, then $A < C$.

4. The shift from evaluating the properties of equality and inequality, using physical objects, to having only *letter formulas* available.[25] The solution of varied problems which require a knowledge of these properties (for instance, problems involving the connection between relationships, such as: Given $A > B$, and $B = C$; find the relationship between A and C).

Topic IV -- The operation of addition (and subtraction).

1. Observations of *changes* in objects in one or another parameter (such as volume, weight, length, or time). Representation of increase and decrease with the symbols "+" and "-" (*plus* and *minus*).

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[24] To explain the curriculum we are using some mathematical terms we did not give the children (we shall indicate the range of terminology used by them in our description of the actual teaching done according to this curriculum).

[25] Independent work with letter formulas is not new. But it was given particular attention here, and it has been systematized and stabilized.

2. Upsetting a previously established equality by changing one or the other of its sides. The shift from equality to inequality.[26] Writing formulas of the type: if $A = B$, then $A + K > B$; if $A = B$, then $A - K < B$.

3. Methods of shifting to a new equality ("reconstructing" it according to the principle: Adding an "equal" to "equals" yields "equals"). Working with formulas of the type: if $A = B$, then $A + K > B$, but $A + K = B + K$.

4. The solution of varied problems requiring that addition (and subtraction) be used in shifting from equality to inequality and back.

Topic V -- The shift from an inequality of the type $A < B$ to equality through addition (or subtraction).

1. Problems which require this shift. The necessity of determining the value of the difference between the objects being compared. The possibility of writing an equality when the specific value of this difference is unknown. The method of using x. Writing formulas of the type: if $A < B$, then $A + x = B$; if $A > B$, then $A - x = B$.

2. Determining the value of x. Substituting this value in a formula (introduction to parentheses). Formulas of the type: $A < B$, $A + x = B$, $x = B - A$, $A + (B - A) = B$.

3. Solving problems (including "word problems") which require the indicated operations.

Topic VI -- Addition and subtraction of equalities and inequalities. Substitution.

1. Addition and subtraction of equalities and inequalities: if $A = B$ and $M = D$, then $A + M = B + D$; if $A > V$ and $K > E$, then $A + K > V + E$; if $A > V$ and $B = G$, then $A \pm B > V \pm G$.

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[26] The possibility of this shift has to do with one of the properties of addition, monotony (in a certain sense this pertains to subtraction as well).

2. The possibility of representing the value of a quantity as the sum of several values. Substitutions of the type: $A = B$, $B = E + K + M$, $A = E + K + M$.

3. The solution of various problems involving the properties of relationships to which the children have already been introduced (many of the problems require simultaneous consideration of several properties and adeptness at evaluating the meaning of the formulas; the problems and their solution are described below).[27]

Such is that part of the curriculum intended to take three and a half to four months of the first semester. Our experience with the experimental instruction indicates that if the lessons are planned correctly, the teaching methods perfected, and teaching aids well chosen, children can master the material fully in a shorter time (three months).

From this point on the curriculum is structured as follows. First the child is introduced to number as the expression of a relationship between the whole of some object and a part of it. The relationship itself and its concrete referent are expressed by the formula $\frac{A}{K} = n$, where n is any whole number, usually taken to within a "unit" (a whole number can be obtained only by choosing the material especially). From the very start the child is "forced" to keep in mind that measuring or counting may yield a remainder, a fact which needs to be especially mentioned. This is the first step toward working with fractions.

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[27] In this variant there is no topic to introduce the child to the commutative and associative properties of addition (prior to the introduction of numbers). This is done in the second semester when the children are working with numbers (writing them both as figures and as letters). In the most recent variant of our curriculum this topic is presented in the "prenumerical" section. Preliminary data from experimental instruction show that it is worthwhile to include this topic and that first-graders are capable of learning it. Since the material in this book was drawn mainly from instruction by an "old" variant, the new topic is not included in it (a special article will be needed to describe how children learn it).

Once the child has learned the above method of obtaining a number it is easy to teach him to describe an object using a formula of the type $A = 5k$ (if the relationship equals 5). Knowing this formula and its equivalent makes possible the special study of the interrelations among an object, its basic unit of measure , and the results of counting (measuring), as well as the preparation of the child to work with fractions (and in particular, to understand the basic property of the fraction).[28]

Another line of development the curriculum may follow in the first grade is to transfer the basic properties of quantity (the disjunction of equality and inequality, transitivity, and reversibility) and of the operation of addition (commutativity, associativity, monotony, and the possibility of subtraction) to (whole) numbers. In particular, in working on a number ray, the child can readily convert a sequence of numbers into a quantity (for instance, by distinctly recognizing the transitivity of such notations as $3 < 5 < 8$ and at the same time making the connection between the relationships "more than" and "less than": $5 < 8$ but $5 > 3$).

Once the child has been introduced to certain "structural" features of equality, he can approach the relationship between addition and subtraction differently. The following transformations, for instance, are performed as one goes from inequality to equality: $7 < 11$; $7 + x = 11$; $x = 11 - 7$; $x = 4$. Or, the child may add and subtract elements of equalities and inequalities, performing oral calculations in the process. For instance, given that $8 + 1 = 6 + 3$ and $4 > 2$; find the relationship between $8 + 1 - 4$ and $6 + 3 - 2$. If this expression is unequal, make it equal (first the symbol for "less than" needs to be put in, and then a "two" added to $8 + 1 - 4$). Thus if the numerical series is treated as quantity, the skills of addition and subtraction (and subsequently, multiplication and division) may be developed in a new way.[29]

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[28] The shift to numbers, the part of the curriculum pertaining to them, and the results of teaching using the curriculum are described in the next section.

Let us compare the outline of the curriculum to the mathematical characteristics of quantity. There will be no direct correspondence between the two because the form in which these characteristics are expressed is governed by the requirements of the "gross" theory and axioms, while the curriculum is designed to perform a number of specific psychological and educational tasks connected with structuring the academic subject, the most elementary section of it, in fact.

The basic task of the section defined by Topics I and II was to teach the child to distinguish parameters of objects which possess three particular relationships. In addition to learning methods of isolating these parameters, he was to learn symbolic means of primary mathematical description of the relationships (letter symbols and formulas). In a series of intermediate stages the child was to structure a special mathematical "object" and proceed to the study of its properties (this object takes the form of abstractly presented equalities and inequalities).

In Topic III, the child was introduced to actual properties of quantities within a particular system for representing them (in formulas of equality and inequality). The child increasingly "divorced himself" from using objects to observing relationships and shifted to verbal and logical evaluations (constructions of the type: "if . . . and . . ., then . . .").

In Topic IV the child learned to observe changes in the specific values of quantities, to compare new values with old ones, to designate the results of this comparison as "increase" or "decrease," to write the results using the symbols "+" and "-," to coordinate them with the properties of equality and inequality, and to go from one to the other by means of addition and subtraction.

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[29] We are speaking here of possible new lines along which the course may develop after the preliminary introduction to the properties of quantities and the operations on them. In our experimental work, not described here, we have in fact already explored many of these possibilities in teaching first- through fourth-grade mathematics.

In Topic V, the child was brought to the discovery that an inequality between quantities may be "taken away" by determining the specific difference between them. The child thus confronted a very simple form of an *equation*. Here, too, a deeper understanding of the relationship between addition and subtraction was acquired.

In Topic VI, the preceding topics were synthesized. It was shown that the specific value of a quantity may be replaced by the sum of several items, that one external form by which a quantity is expressed may be replaced by another (substitution), and so on, all of which lays the groundwork for an introduction to the commutative and associative properties of addition.

From this comparison it may be concluded that our curriculum, designed as it is to perform certain psychological and educational tasks, contains information about the basic properties of quantities as indicated by the axioms of mathematics. At the same time, in setting up this curriculum, we were introduced to the concrete problems of *projecting* scientific knowledge onto the plane of an academic subject, and we do mean introduced, for these problems need further experimental and theoretical study. In particular, there is the problem of finding the most suitable way of introducing the child to the realm of "comparable elements," so that he will be able *to combine* it with and *to differentiate* it from the realm of "incomparable elements" according to certain attributes, distinguishing the *relationship of order* and correctly correlating it with the structure of an operation such as addition. This problem touches directly upon the ways of structuring the elementary section of the school subject of mathematics, inasmuch as the child's most general orientation to the mathematical side of reality comes from the very "heart" of this section.[30]

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[30] As these problems are solved, it is apparent that the emphasis in mathematics instruction will shift from "techniques of calculation" to the study of the *structural characteristics* of mathematical "objects." A different academic subject thereby will take shape from the present one which mainly prepares the pupil for the further study of mathematical analysis (in the preceding section of the book we discussed some of the bases for such a subject).

The Organization of Instruction According to the Experimental Curriculum

One feature of our research is that the curriculum, with all its "components," has been devised on the basis and in the course of specially organized experimental instruction. Each time a psychological or educational problem of curriculum design arose, we attempted to solve it in the subsequent school year when new experimental classes were using new variants of the curriculum. Special attention was devoted to studying the child's own system of learning the material, and to developing research methods for determining the extent of this learning. Attention was also devoted to studying the possibilities for later "use" of the knowledge acquired, and mainly, to studying the nature of the pupils' thinking both in the course of the academic work and in the solution of varied test problems (in the classes and individually).

During the school year, 1960-61, one first-grade class at School No. 91 in Moscow (E. S. Orlova, teacher) was taught using the first variant of the curriculum. The following year, 1961-62, four first-grade classes used a different variant of the curriculum (at School No. 91, with V. T. Mikhina, teacher; in two classes of School No. 11 at Tula, with T. A. Frolova and M. A. Bol'shakov, teachers; and at the school in the village of Mednoe, Kalinin province, with A. I. Pavlova, teacher). Meanwhile, the previous experimental first grade used a special curriculum for the second year of instruction.[31] Many of the specific topics of the curriculum and the organization of school time were determined more precisely during this year, and the basic difficulties that the children and the teacher had were clarified. Detailed lesson plans for the entire first grade were compiled with consideration for these matters.

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[31] The majority of the experimental classes which used our curriculum in the first grade continued in the second, third, and fourth grades (and in 1964-65, the fifth as well) to use special curricula which differed substantially from traditional ones (letter symbols were used "routinely"; negative numbers and fractions were introduced in the second and third grade; the system of coordinates in the fourth, and so forth). As it would be a separate task to describe the whole mathematics curriculum for the primary grades, we shall simply note that the groundwork laid in the first grade was built upon in the second through fourth grades and at the same time the construction of the foundation was improved from the "elevated" vantage point of these classes.

Four classes were taught in 1962-63 on the basis of these improvements (at School No. 91, with A. A. Kiryushkina; at School No. 11, with A. P. Putilina; at the Mednoe village school, with M. I. Dem'yanenko, and at the school affiliated with the pedagogical academy at Torzhok, Kalinin province, with T. B. Pustynskaya). Five classes were involved in 1963-64 (two at School No. 91, with T. G. Pil'shchikova and V. A. Vvedenskaya; and one each at School No. 786 in Moscow, with G. G. Mikulina; School No. 11, with V. P. Polyakova, and the Mednoe village school, with Z. N. Nemygina). And finally, there were three classes in 1964-65 (two at School No. 91, with E. S. Orlova and G. V. Chernysheva, and one at School No. 11, with O. P. Filatova).

Thus, in five years seventeen classes in both city (Moscow, Tula, and Torzhok) and rural (Mednoe) schools were given experimental instruction according to our curriculum for the first grade.[32]

Elementary school teachers did the teaching in all the classes. The majority of them had a secondary pedagogical education (while some had college training). They had from three to fifteen years of experience. As a rule, these were skilled teachers who knew the traditional curriculum and methods well and became "used to" the new demands in the course of the experimental work itself.[33] There was nothing unusual about the makeup of these classes. They consisted

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[32] In 1963-64 and 1964-65, our curriculum was used for first-grade mathematics teaching at School No. 82 in Khar'kov (with F. G. Bodanskii and V. S. Kruglyakova, teachers). In 1964-65, mathematics teaching in several first-grade classes of the experimental school No. 52 at Dushanbe was based on it (with M. N. Vasilik, teacher).

[33] We express our sincere gratitude to all the teachers who taught by the experimental curricula, for their readiness to become involved in something new and for their constant aid in solving many problems of organization and methods.

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of the children living in the school neighborhood, with no one excluded (the number of pupils in the various classes varied from thirty-two to forty).

Our research was aimed at tracing the way the curricular material was learned and the thought patterns that occurred in the learning process. We used several methods: (a) systematic observation of teacher and students in class; (b) analysis of students' performance of daily class work as seen in their notebooks; (c) analysis of results from special tests; (d) special individual checking of students' knowledge of particular topics of the curriculum, as well as the nature of their thinking.

Observing classes and analyzing the daily performance of assignments helped us assess the *dynamics* of the work being done by teacher and children at a given time, which is the key to describing the learning process. We devised special tests which would reveal not only whether the children had learned the material but also the degree to which they really understood it. In addition to familiar types of exercises, these tests included problems in which mathematical relationships the children already knew were expressed in an unfamiliar form for the first time. To solve these problems it was necessary to have a real understanding of the material and a grasp of the *consequences* of certain relationships. In some instances the students were given *particularly difficult* problems so that we could judge by the way they solved them what the "ceiling" of their understanding of the matter was.

Individual investigation of what the students knew and how they thought was of special importance. It took two forms: (a) solving difficult problems whose basic content coincided with the material dealt with in class (Here we were checking particular aspects of the students' approach to mathematical problems. It is difficult to assess these matters when the whole class is taking a test); (b) performance on a special group of exercises not directly related to the material already dealt with but through which one may judge the nature of each student's inner (mental) plane of operations.[34] (Since these test exercises and problems bear a close relationship to the instructional material itself, they shall be described as we give the results of the teaching done by the experimental curriculum.)

Considerable experimental material has been gathered in the course of our five-year investigation (published in part in our study [3], as well as in articles by the teachers, T. A. Frolova [7] and A. A. Kiryushkina [11]). We have concentrated on the most characteristic features of the teaching process and its results, the features which are typical of the various classes. These are the features which will be primarily described (naturally it will not be feasible to talk in detail about the characteristics of a particular class). At the same time, in addition to giving summary data, the results of the instruction in two or three classes will be traced which were observed and investigated with particular care.

Characteristics and Results of Instruction by the Experimental Curriculum

We shall subdivide the data according to the main "steps" in the teaching, and describe, under each topic, the way the material was learned.

Topic I -- (comparing and assembling objects according to various parameters).

Even before children start school, they have faced practical problems of comparing things according to different physical parameters (length, volume, and weight, mainly). At home or in kindergarten they have drawn pencil lines of equal length, for instance, have cut circles of the same diameter (or area) out of paper, or have made identical "cakes" (of equal volume) out of wet sand, clay, or plasticine. Many children are familiar with weighing before they start to school, since they have observed salesclerks at work. They have also faced the problem of assembling things according to a model, in one form or another (picking out blocks or doing appliqué work, for instance).[35]

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[34] Ya. A. Ponomarev has made a systematic study of our students' mental plane of operations (the final chapter of this book describes the methodology of this study and some of its conclusions).

We have observed that most urban and rural children not only are familiar with these practical problems but have already learned some general methods of comparing things by length, volume, weight, and composition (such as superimposing on material a model of a shape to be cut out, or holding the edge of a block up next to a piece of plasticine as a way of comparing their volume). Many children know the words "length," "weight," "quantity" (in the sense of volume), and of course, "more than" and "less than" and analogous relationships such as "longer" and "shorter," and "heavier" and "lighter." Thus as a rule, by the age of six or seven children have a practical grasp of certain quantities, they can distinguish relationships of the type "more than" and "less than" and use words to designate them, and they are guided by these relationships in solving problems involving the comparing and assembling of objects.[36]

The goal of Topic I (which lasted six hours) was to discover and, more important, to systematize the children's notions about methods of comparing things, and to teach them to make rapid and accurate associations between certain terms and such parameters as length, volume, and weight.

First the children were given the task of selecting, from the available objects, an object of "the same" length, volume, or color

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[35] By assembling we mean putting the component parts of a thing together, after having selected them from some other material.

[36] A study made recently by L. A. Levinova showed that many children of five and a half to seven years of age are able to distinguish a property such as transitivity relatively well and can focus on it in solving various problems -- in particular, those in which the elements being compared are given only orally or are designated by symbols which have been agreed upon (such as objects a, b, and c).

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as a model. The model might differ from the other objects in some of its properties (for instance, when length was the criterion, it might be of a different color or material). It turned out that initially almost half the children tried to pick out objects which were identical to the models not only in the attribute indicated, but in other attributes as well. For instance, if length was indicated, the children tried to find an object which matched the model in its color, material, and other properties. These children actually knew how to focus on a particular attribute, of course, but they still could not abstract it from other properties not mentioned in the problem, when instructions were given orally (some of the children did handle this successfully).

But by solving special problems, all the children quickly learned to choose objects according to a single attribute. The same object (such as a strip of paper) could be the basis for the selection of various objects (some by length, others by color, and so forth). In the course of this work the children gained practice in such skills as superimposing one object on another (selection by length), putting the edges of blocks together (in selection by volume), and so forth.

The next problems required the children not simply to choose but to make a new object which matched the model in a particular attribute. As they worked with strips of paper, small sticks, blocks and plasticine, water and weights, they learned to perform such tasks as making a piece of plasticine of the same volume as a block (children usually call volume "size"), cutting a strip of paper the same length as a stick, matching weights, and so forth. They were given special exercises to introduce them to "making" an object out of its component parts. It was pointed out to them that things can be matched by their duration or by their volume).

Understandably, this matching was done by the simplest practical methods of direct sense perception (eye, ear, etc.). In some cases (involving length or weight, for instance) the match with the model was relatively precise, while in others (such as making a plasticine block) it was hard for the children to achieve the desired "fit" with the model, and they were quite aware of the possibility of greater accuracy even though they could not actually attain it. It is striking that many of the children grasped the conditional, approximate nature of this match. In fact, some openly said that "you can't do it exactly anyhow" by eye, that "you'd have to have a special kind of machine" (statements by Dima K., Tolya V., and others). Some children, however, evaluated the results of the matching categorically and "absolutely" ("I made a stick just like it"), if they did not see the practical possibility of further improvement. If they noticed a discrepancy or "sensed" that their object might deviate from the model, they would agree with the teacher about the possibility of "perfecting" the object in principle ("But it's supposed to be very exact -- I'll try to do it. . ." said Nadya D., from Moscow).

The teachers indicated to the children that the matching they were doing was approximate, that deviation from the model might occur but that it should be as unnoticeable as possible. It is important to note that the children understood this "limitation" which forced them to qualify somewhat their statement that their object matched the model ("We could say that the block is the same volume, but is is a tiny little bit different. . ." was the view of Zhenya T. from Mednoe). At the same time when some of the children (generally about a third in each class) were asked directly, "Might there be a difference here? Look carefully!", they attempted not only to find it but to "remove" it. But if they did not notice such a difference or noticed it but could not eliminate it, they hesitated about whether the object and the model could be considered equal ("I don't know. . . they might be equal. . ." Lyuba V. from Moscow said; "They're supposed to be equal . . ., but I don't know if they are," said Vanya O. from Moscow).[37]

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[37] During the work on Topic II all the children become distinctly aware of the practical necessity for tolerating possible imprecision, and the conditional nature of statements about the equality of objects.

Using objects as aids to examine the relationships of equality and inequality assumes that the children are capable of "separating" the directly observable properties of the objects from certain theoretical assumptions made in discussing these properties. This interesting matter needs further study since it concerns the child's developing theoretical judgment and his understanding of conditionality and assumptions, counterbalancing direct observation.

The work done with all the experimental classes shows that children have no difficulty learning the material in Topic I (within the limits indicated in our curriculum). After five or six lessons nearly all could select or make -- within practical limits-- an object which "matches" a given model. They could use the same thing for various models if the various parameters of it were indicated to them. By this time all the children clearly and rapidly associated the terms "length," "volume," and so on with the corresponding aspects of the things.

Topic II -- (comparison; letter formulas for equality and inequality). Comparing and assembling are practical operations which result in new things (such as a board equal in length to the model, or a weight as heavy as another). Comparing objects for an attribute, however, is a theoretical operation. It results in knowledge about a particular type of relationship between objects. A course of practical operations may be charted on the basis of it.[38]

In Topic II, children were introduced to the comparison of objects according to particular parameters, where three forms of relationships were distinguished and the results of the comparison recorded in a letter formula. First the children were to determine whether the material (strips of paper, sticks, etc.) would be suitable for comparing with the model. In some cases they found that the material would do -- and what is more, that "nothing had to be done" to it since it was "already just like the stick" (the model, that is). In other cases the material would not do -- it was "shorter" or "smaller."

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[38] Using objects to make comparisons does not change the theoretical nature of the matter, for the result of this operation is particular knowledge rather than a thing made (or picked out).

The results of a comparison were usually formulated verbally: "This board is the same length as that one" (with the boards being held up); or "there is less water in the mug on the left than in the one on the right." Some children made this kind of distinction: "These are equal but these are not." The teacher would "accept" such answers but immediately demand that they be made more precise -- what else could be said? The pupils rapidly found the answers (longer, lighter, and so forth).

The teacher would say to the pupils: "Look around and find objects which are equal (or unequal) in some attribute" (the children understood this term well). Some pupils were able to point to the windows in the classroom: "They are equal in size" (they meant area). Others would hold up pencils: "The red pencil is longer than the blue one." As a rule, many answers were given.

At this point the children used the words "more than" and "less than" most often only to refer to volume. In other cases they gave the direct qualitative characteristics: thicker and thinner, longer and shorter, heavier and lighter, and so forth. They needed special work in order to be able to "reduce" all these characteristics to the abstraction "more than" and "less than." It was carried out gradually and in several stages.

First the children would determine, on their own, the attributes by which certain objects could be compared. We shall quote from the report of a lesson on September 8, 1963 (in Moscow) where this work was being done. The teacher showed the children two weights (one black and one white) and asked by what attributes they could be compared.

Pupils: They can be compared by weight (they point to the scales), by height, or by their bottom (they mean the size or area of the base).[39]

Teacher: What might you say?

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[39] These answers were given by individual pupils, of course. Here similar answers which came in succession are combined (this is to shorten the description of the lessons). Here and below, typical answers will be indicated under the heading "pupils" (while in other instances we shall quote answers given by particular pupils).

Pupils: They are unequal (in weight or height).

Teacher: How can you express this more precisely?

Pupils: The black weight is heavier, higher, bigger, thicker than the white one.

Teacher: What does that mean -- heavier? That the black weight weighs less than the white one?

Pupils: (They laugh.) No, not less, but heavier. . . It weighs more!

Teacher: The white weight is lighter -- how else might you say that?

Pupils: (About half the class raise their hands.) The white weight is less, lighter in weight than the black one.

Analogous work is done on other attributes as well, with the teacher supplying leading questions. Along with the teacher the children establish that "heavier" is more in weight, "longer" is more in length (or "height" or "stature"), "harder" is more in hardness, and so forth (and correspondingly for "less"). In connection with this, the teacher gives the children various problems requiring this kind of "deciphering."[40]

Special attention is drawn then to the fact that such words as "longer" and "heavier" in themselves tell what attributes are being compared (when the children are given problems using these words, they find the necessary objects). But if the words "more" and "less" are used, one must note in addition what attribute is being compared (such as weight or area).

The concluding stage of this work was to point out that if it is possible to find the attribute by which the objects are being compared, then they will be either equal or unequal, written using the special symbols "=" and "≠." But the latter symbol can itself be made more precise. With inequality, one object is less than or more than the other (in the particular attribute), written using the symbols "<" and

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[40] Classwork consisted of specific problems the children were to solve by working on their own with objects, observing operations by the teacher or other pupils, and seeking and formulating verbal answers. These problems are presented in the summary of the lessons. Since it is not possible to quote them fully here, in a number of instances we use such expressions as "the children establish" or "it is pointed out to the pupils" to designate these problems and their solution in abbreviated form.

">." The children learned to use all of these symbols to record the results of comparisons. They also performed the "reverse" tasks. Guided by the written symbols ("<" or ">"), they would select the most diverse objects to illustrate the relationships indicated -- blocks and mugs (for volume), squares and triangles (for area), and bars (for weight). (The work being discussed involved special teaching aids. In actuality, however, problems were constantly being given, both at this point and later, which required the children to find these relationships in the real-life objects around them.)

A problem arose here in that relationships had to be given according to the special rule "from left to right" ("this is less than that" -- from left to right). From five to seven children (out of 32 to 37) in each class needed special instructions from the teacher and a number of special exercises in order to learn the "direction" of the comparison. The rest of the children mastered this point completely after one or two explanations.

As mentioned earlier, Topic I included problems of assembling as well as ones of comparing. This practical activity also has its theoretical parallel, a _special form of comparison_ which was demonstrated to the children and which was of great importance later when they came to numbers. Inasmuch as this form of comparison is quite out of the ordinary and the way its results are evaluated is unusual, it would be best to digress from our presentation of the instruction process and describe it briefly.

Imagine that a group of children are to be given a pencil apiece (this is a problem of assembling in which the model of the complement is "a child with a pencil"). One must first determine the relation between the group of children and the group of available pencils -- that is, determine whether there are _enough_ pencils.

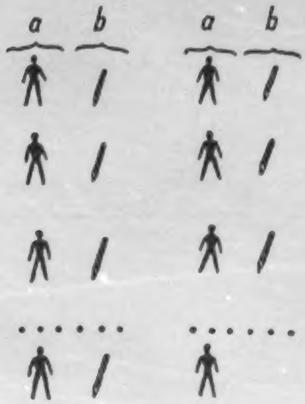

Figure 1. -- Comparison of aggregates of objects.

This is a comparison problem. The method of solving it (without using numbers) is obvious -- each pencil is given to one pupil; a "one-to-one correspondence" is established (Figure 1). Three different answers are possible. The groups are equal, there are more pupils than pencils, or there are fewer pupils than pencils. Before the comparison was made, the criterion for it was indicated (each pencil is put with an individual pupil, which is the requirement that follows from the model). The groups of "objects" have been turned into quantities (see Kagan's definition, p.127).

This particular comparison is striking in that the criterion for it is the juxtaposition of two physically distinct objects. But an important matter has been left out. In actuality when a practical problem is solved the objects have to be assembled according to the most diverse criteria, with comparison of the objects separately being only a particular instance (the very concept of "complement" indicates this). A whole group of objects from one aggregate may go with (correspond to) one physically distinct object from another aggregate. The correspondence is determined in each instance by the actual situation and the characteristics of the complement which, since it is the model, dictates the criterion for comparison.

Apparently because of the specific nature of these criteria, they have not become generalized under a "single" parameter such as "recognized quantities" have, with their special designation such as "length" or "hardness" (see Kagan's ideas about the way these designations have come about [10:106]).

Here is an example of the comparison of elements of conditional aggregates (Figure 2). The model of the complement (a) and its component "parts" (the aggregates of "thick" (b) and "thin" (c) bars) are given. The groups of "parts" are to be compared according to the criterion contained in the model (a group of three "thin" bars corresponds to a "thick" bar). One may conclude from comparing them (d) that, first, the aggregates are unequal, and second, the left one is greater than the right one (with respect to the criterion given).

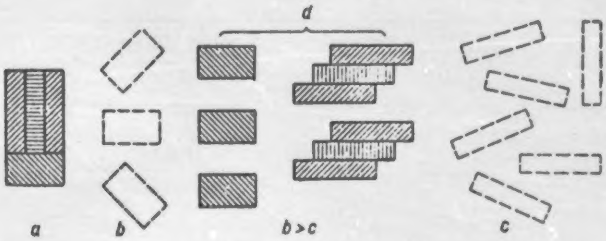

Figure 2. -- Comparison of aggregates of objects according to the criterion given by the "composite element": (a) is the model of the complement; (b) and (c) are parts of the complement (their position before they are compared); (d) is the comparison of the parts and the results of it.

The principle of "one-to-one correspondence" has not been violated, for the very method of comparison, the very operation required to satisfy the criterion, "shaped" the groups of physically distinct objects into abstract elements (indicated by brackets in the illustration). But a "distinct" element which is formed by an operation in fulfillment of some criterion cannot be identified with any physically distinct thing at all.[41]

Aggregates of discontinuous objects based on some criterion determined by the characteristics of the model of a complement may thus be turned into a quantity. At the same time it should be emphasized that this will be a quantity of a special kind not identical with the physical quantities usually designated by this term.

There were problems in Topic II which required the pupils to compare aggregates of objects, with a number of instances in which "groups of elements" were to be compared (the characteristics of the model of the complement determined this). The children solved all the problems involving comparison of distinct objects rapidly and with almost no errors. They would confidently place the objects in vertical "columns" (with the teacher showing how they were to be arranged) and, after comparing the objects horizontally, would formulate the answer orally or would record it with a symbol (equal to, more than, or less than).

Problems involving "groups of elements" caused many children[42] difficulty. For instance, they were told to choose "bricks" to build a "little house." Each "house" was to consist of a big block and several small ones (Figure 3a). They had to sort out the available material for each house and compare the groups of blocks (they readily understood and used the word "group"). When given the problem,

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[41] Of course, the theoretical mathematician working at the level of concepts has in mind an abstraction having any concrete physical meaning rather than something "physically" distinct. Unfortunately, in the area of mathematics teaching methods, this abstraction or abstract distinctness is identified with actual physically distinct things. In our view this leads to serious difficulties in teaching.

[42] Our general quantitative terms have the following meaning (here and below). "Many children" means about two-thirds of all the pupils in the class (out of 32 to 37); "the majority of the children" means more than two-thirds of the pupils (27 to 31 out of 32 to 37); "almost all the children" means the pupils in the class with the exception of one, two, or three.

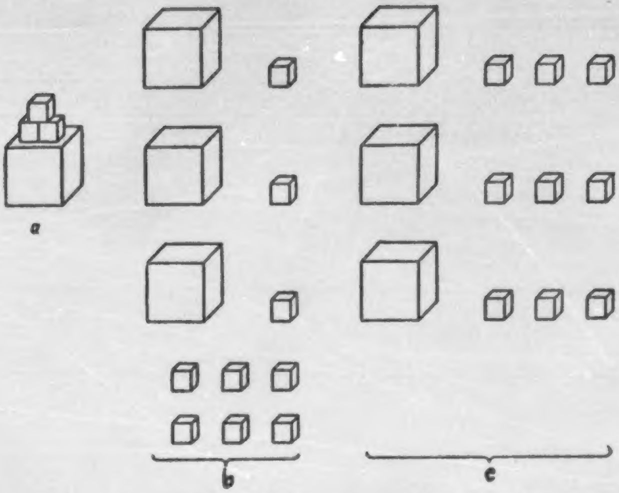

Figure 3. -- Diagrams of incorrect (b) and correct (c) utilization of a criterion provided by a model of a complement (a).

many of the children arranged the blocks one to one (Figure 3b). It turned out that there were fewer big blocks than little ones. But some of the children laid them out correctly and obtained an equality (Figure 3c). The teacher would juxtapose these answers and, aided by the children, would explain the causes of the difference between them, and then would draw attention to the model. Then for the purposes of demonstration he would solve an analogous problem (with the children observing). He would use the same collection of objects (such as the blocks) but a different model, and the results of comparing the groups would differ. This would demonstrate to the children again and again that in making such a comparison, one must always know and remember why the objects are being "selected" and what needs to be "looked at" or "remembered" in order to make an accurate comparison.

It should be noted that once they had correctly arranged the "groups of elements," the children had no particular difficulty determining the actual relationships of equality and inequality.

The question accompanying the problem was usually phrased this way, "What can we say about the left and right groups if we need to find out whether there are enough of these materials (bricks, balls, and so forth) to . . .?" (A description of the complement would follow here.) As a rule the children would answer correctly, "There are enough -- the left group and right group are equal," or "there aren't enough -- there are more on the left." There might in fact be fewer separate elements in the left group here than in the right one. The point was that the comparison was being made according to a particular criterion.

Some children experienced new difficulties in certain conflict situations. For instance, G. G. Mikulina (of Moscow) gave the children the following problem in the twelfth lesson. She drew some mugs on the board.[43] The children were to make a copy in their notebooks in the same order and, after comparing them to a given complement (Figure 4a), were to record the result. Many of the children, correctly following the criterion given (a little mug goes with each big one), wrote an "equals" sign. But some confidently put the symbol for inequality (≠) and, to make its meaning more specific, put in the symbol for "less than." This was their reasoning, "The mugs on the left will hold less than the ones on the right, so I put the sign for 'less than'" (Serezha R.).

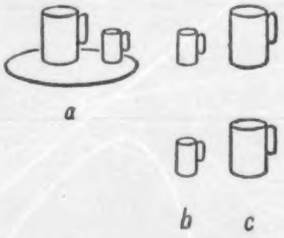

Figure 4. -- Diagram of the "collision" of criteria of comparison: (a) is the complement model; (b) and (c) are the aggregates being compared. In a comparison by volume, b < c; in a comparison according to the complement, b = c.

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[43] In the tenth lesson the children had begun to draw sketches of the objects being compared; the method of transition is explained later.

Thus these children were comparing the mugs not according to the complement but by volume. For them the simpler and more familiar criterion "won out." Using this example, the teacher showed the pupils that it was possible to compare the same objects for various attributes, and she emphasized the importance of keeping them carefully in mind when working with them (all the children solved the next "conflict" problems correctly).

At first glance these problems seem artificial and unnecessary (as we have been told upon occasion). But as we see it, such an attitude is a manifestation of an unwillingness to "rummage" in the sources of mathematical assumptions about which Kolmogorov was speaking and which, unfortunately, one still finds. Of course a person who has already learned an abstraction which embodies certain assumptions, and who is accustomed to using numbers (and can even divide them) "cracks" these problems "like nuts." But the child has not learned such criteria. For him they all need to be deduced. As a part of such learning he must be shown the difference between the immediate characteristics of things and the approach to them from the standpoint of mathematical problems.

Thus, the essence of a mathematical problem does not change as the criteria of comparison are changed. And this is the very point which needs to be made plain to the child in demonstrating the possibility of changing the criteria for the same objects. In this process it becomes clear that even though the specific form of a relation may change (equality being replaced by inequality), the operation of comparison itself remains the same, overruling the customary direct evaluations (a group is defined as "larger," for example, even though it might have fewer individual elements in it than the "smaller" one). Furthermore, working such problems undoes the tendency to evaluate the relations between objects from the point of view of one particular abstract illustrative case, when a "one-to-one" correspondence is identified in advance with the direct correspondence between specific things.

The children themselves enjoyed solving problems involving a change in the criteria of comparison (with both discontinuous and continuous objects). E. S. Orlova's class (in September, 1964) was particularly lively during this phase. The children solved "hard" problems with interest and carried on an intelligent discussion about the reasons why the form of relationship changes during comparison. They learned to orient themselves carefully to the criteria indicated or implied (through relying on the model), as was evident in their performance on difficult tests. And in the second semester, when they went over to using numbers, these children had virtually no difficulty evaluating the numerical characteristics of an object from the standpoint of *any or a changing base* of counting (especially with a "composite" base). From these data it may be concluded that the direct relationship to a separate object as if it were an absolute brick for building mathematical models was "undermined" from the very beginning in our children.[44]

One of the main items in Topic II is the representation of relationships using formulas. The shift to this representation is achieved through two intermediate stages -- first a "copies" drawing, then an "abstract" representation with lines. In the ninth or tenth lesson the teacher asks the children to solve problems using objects drawn on the board (mugs, blocks, and various "parts" to be assembled such as a "bicycle" and its "wheels"). These drawings are substitutes for real objects although they resemble them. True, there is much oversimplification. A block may be represented by a square - that is, by just one of its surfaces. The children make corresponding drawings in their notebooks, find the relationships between the "objects" in them, and put in the symbols.

With this method of working, the children can make abstractions from the immediate, material "texture" of the objects they are comparing and more distinctly isolate the criteria of comparison in them. And it becomes easier for the teacher to select problems, since the most diverse objects can be shown in drawings. Working with the drawings does not in itself cause any particular difficulty. The children transfer the methods they had used in comparing the objects (Figure 5).

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[44] Again we should mention that "separate object" is not identical with the concept of a "separate element" (of a set) in the abstract mathematical sense.

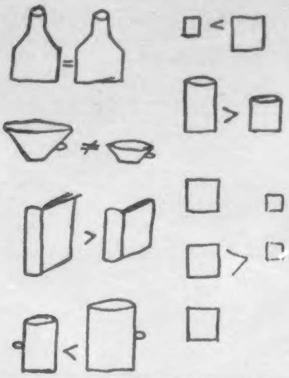

Figure 5. -- Representing the results of comparison in a copied drawing (notebook of Olya U., a first-grade pupil in Moscow).

At the same time the conditional nature of the connection between the statement and symbol for the relationship of equality, and the representation of it in a drawing, becomes particularly apparent at this stage, since the lines, squares, and circles are "equal" only in a very approximate way (all the more so because the children usually draw haphazardly). In September, 1962, we tested the children's attitude toward this fact. In individual conference each pupil in a Moscow class (there were thirty-two in all) was shown a large or small inaccuracy in a sketch he had made representing an equality. In answer, twenty-one of them immediately cited the symbol ("But I have an equals sign there, so that means they [the squares] are equal," was Tanya Z.'s answer). The other eleven started to "improve on" their sketch, attempting to make the elements of it as nearly equal as possible. Then eight of these pupils also referred to the symbol (although four of them had not put it in), and three considered the improved representation itself "convincing" enough to demonstrate the equality of the objects.

Thus the majority of the children in this class (as in the others, by the way) were guided basically by the assertion of equality and by the symbol for it, and not by the representation of this equality in a drawing (which is itself a symbol).

In the first variants of the curriculum, the shift to letter symbols came immediately after this stage. It proceeded satisfactorily on the surface, but special checking showed that many of the children then had difficulty "interpreting" the meaning of the letter designations. In subsequent years, therefore, another stage of work between the "copied" drawing and letter formulas was included: using _the relationship between lines_ to represent the results of a comparison of objects by any criteria at all.

The necessity of using such a means became apparent at the preceding stage when the results of comparing the heaviness of weights or the loudness of sounds could not be depicted in a drawing. The teacher would take the verbal definitions of the relationships the pupils had found (such as "heavier," meaning _more than_, or "louder," meaning _more than_, and so on), and show that these results can be "written" using _lines_. The relationship between the length of the lines corresponds to the relationship between the objects according to the given parameter (such as weight or loudness).

The following quotation is from the record of a first-grade class at the Tula school (V. P. Polyakova, teacher).

Teacher: The weight on the left is heavier . . . (points to the scales). What is another way of saying that?

Tolya S: It (the weight) . . . weighs more than the other one.

Nina K: The weight on the right weighs less.

Teacher: Right. The weights look as if they are identical but they differ in weight. How can we write this, mentioning the weights and what we found out about them? . . . Let's write down our result using lines -- here, I'll draw them -- one on the left for the left one, and the right one on the right. I'll make them the same length, since one of the weights weighs less than the other . . .

Pupils: (Many of them raise their hands immediately; there is a buzz of astonishment.) Not that way: The weights don't weigh the same but the lines on the board are the same length. They shouldn't be equal.

Teacher: Then what should I do? Can I use lines to show what the weights are like, or not?

Pupils: You can! But not that way!

Teacher: Then how? Who can do it?

Pupils: (Several hands go up -- ten or twelve out of thirty-four.)

Teacher: (Calls three pupils to the board.) You each do it as you think it should be done. The others draw it your own way in your notebooks.

The pupils did this (the drawings expressed the relationship correctly), and then they discussed the results with the rest of the class.

Lines were then used in this lesson to represent the relationships "more than" and "equal to" in weight, and to represent all three relationships in comparisons by volume, by the duration of a sound uttered, and by the composition of groups of objects.

The teachers gave the children "reverse" problems as well. Going by lines drawn on the board, they were to select objects which would yield this result if compared. As they discussed possible errors, the children were repeatedly made to realize that only the relationship between the lengths of the lines mattered in recording the results this way -- and it was to be the same relationship as the one yielded by the comparison. A series of drawings appeared in the children's notebooks done with colored pencils. In recording the results of the same comparison different children drew their pairs of lines of different "sizes." The teachers took the following approach at this point. They would show the class notebooks in which the pairs of lines were different in length. The question would be posed, "Are these drawings identical or not?" A discussion would begin, and the children would establish that the "drawings" were identical since each pair of lines showed the results of the comparison accurately, and that they were "about the same thing." And several more such "clashes" between the meaning and the external appearance of a notation were set up (Figure 6a).

Several lessons later the pupils were given an unexpected problem. They were to use circles instead of lines to record the result of a comparison of any two children's height. Could this be done? Many of the children thought so and wrote the answer in their notebook

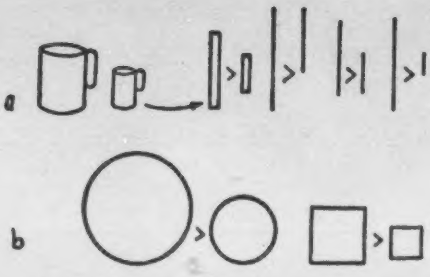

Figure 6. -- Representing the results of comparison through the relationship between lines, circles, or squares: (a) comparison by volume (each pair of lines is identical to the others in meaning); (b) comparison of sounds by loudness and by duration.

or on the board with no help. As a rule, the relationship between the areas of the circles would correspond to the results. The teacher would show the children that squares or triangles could be used to record the same thing and that what was important was to make the relationship between their "size" (area) the same as that in the comparison. This activity interested the children greatly. They were particularly excited about problems in which the results of a comparison of the loudness of sounds, for instance, could be recorded by any means other than the "customary" lines. The pupils would use circles, triangles, and squares which, in the relationship between their areas, accurately depicted the relationship between the loudness or duration of two sounds (Figure 6b). The majority of the children could correctly explain the meaning of what they had put down, the connection between what was being compared and how it was represented, and also the fact that these differed completely except for the matter of "more" and "less."

This phase of the work thus introduced means of transcription whose physical characteristics had nothing in common with the characteristics of the objects being compared (such as the loudness of a sound being depicted by using squares). The possibility of such a transcription is determined solely by the isomorphism of the relationships of equality and inequality themselves, which actually reveal their pure form through such "transformations" and become a subject which can be dealt with later.

This stage of the work was of great importance to the children. In the first place, it clarified and justified for them the very possibility of representing "all in one" this way. And in the second place, many of them, when interpreting a relationship given in a symbol, now tried not only to select actual objects (such as sticks or blocks) but also to represent the relationship "rapidly" in a symbolic drawing in their notebook (lines being drawn or squares sketched to correspond to the symbol given). What becomes central for the children is the relationship itself, its type, rather than the objects through which it may manifest itself.

On this basis a new form of transcription -- using *letters* -- was introduced (in the fifteenth or sixteenth lesson). The direct introduction of letter formulas in these lessons was preceded by preparatory work meant to make two matters clear to the children: (a) the results of comparison by a single attribute may be recorded using *different* "signs" (lines, squares, circles and symbols), and (b) these signs tell about the *weight*, *volume*, *hardness*, or other attribute of one object *in comparison* (precisely this: in comparison) with the weight, volume, or hardness of another object or objects. These matters were usually studied by recording the results of various comparisons of a metallic weight and a block of wood (the weight was heavier but of less volume).

The teacher would give the pupils "freedom" in the choice of signs and then, holding up their notebooks, would show that different children had different signs (some had lines, others had circles, and so on). "You can do it this way," of course, but it is better to choose a sign which is uniform and constant for everyone. As such a sign, the teacher says, people have chosen the letter. If, for instance, a weight and a block are being compared for heaviness, the heaviness of the weight may be designated by the letter A, and that of the block, by the letter B (the teacher writes A ... B on the board).[45] But these letters are equal in "size" and are different in this way from other signs such as squares. What shall we do? How are we to read what we have written if we know that the heaviness of the weight is greater than that of the block? The teacher guides the pupils toward the goal by saying, "The heaviness of the weight is A and that of the block is B," and a substantial number of the children could continue on their own. They formulated the answer orally first: "The heaviness of the weight is A, and it is greater than the heaviness of the block, B."

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[45] The actual work of teacher and pupils in these lessons is only summarized here.

With the children's participation, the teacher would establish that the symbol for "more than" was lacking, so they would immediately put it in. They would thus obtain the formula $A > B$. This transcription would be decoded again, the children taking turns explaining its meaning: "The heaviness of the weight, that's A, is more than B, the heaviness of the block." The teacher then replaced this pair of objects with a new pair to be compared -- a new weight and a block, preserving the relationship between them but differing from the former ones in size and color.

Teacher: What results do we get from comparing these objects by weight?

Pupils: Again the weight is heavier than the block.

Teacher: Now you know a new sign to use to record the results of a comparison. Well, see if you can use it. How do you write the weight of this weight? The weight of the block? Let's write it.

Pupils: With the letter A and the letter B (following the teacher, they write A ... B in their notebooks).[46]

Teacher: Does what you have written tell us everything already?

Pupils: No! This tells about the weight here . . . but we still need something about the results. . .

Teacher: What do we know about these results? How should we record them here when we have the letters? Try to do it by yourselves.

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[46] At this point they had only begun to read and write, and of course in mathematics classes the teacher relied on their preschool experience in writing "printed" letters.

Many of the pupils, going by the preceding transcription, put in the symbol accurately between the letters: A > B; but several put it a line lower, although they were able to give an accurate explanation of what they had written.

The teacher checked the work, again demonstrated the rules of transcription and the proper places for the symbols, and asked about the meaning of the formula and of each of the signs in it.

Teacher: We read it this way, children: A is more than B. But what is A and what is B? What does what we have written tell us?

Pupils: It says we have compared the weight and the block for heaviness: The heaviness of the weight is A, and that of the block is B. The weight weighs more than the block. The weight of the weight is more than the weight of the block. This is written: A is more than B.

The teacher could substitute a new pair of objects and again compare them by weight, but this time the weight could be lighter than the block. The formula A < B would be written down and its meaning interpreted. Then the same objects could be compared by another parameter -- volume. The teacher would emphasize here that the objects were the same but the attribute by which they were being compared had changed. At first the children, working orally with assistance from the teacher, would find that the weight is less in volume than the block.

Teacher: Before, you used to record such results this way: with the line on the left shorter than the right one (he shows them). But now we know another sign -- a letter. If we designate the volume of this weight by the letter A, then how might we designate the volume of the block?

Pupils: By the letter B (however, several of the children begin to show initiative and suggest a different letter -- C, D, or E).

Teacher: Good. Write it this way: A . . . B. What is A and what is B?

The pupils answer correctly.

Teacher: But we can use a different letter to designate the volume of the block: Someone has already suggested D. Let's write it underneath: A . . . D. Have you done it?

Pupils: There isn't any symbol (they put a symbol in both formulas: $A < B$, $A < D$).

Then the teacher questions the pupils to clarify with them the meaning of the formulas, and to establish that these formulas are saying the same thing: That the volume of the weight is less than the volume of the block (A *is less than* B; A *is less than* D). As he does this, he constantly reminds the children that the letters are "talking" about the attribute under comparison: the *heaviness* of the weight, in the one case, and its *volume* (or *hardness*, or *height*, and so on), in the other. But the letters do not in themselves register the results of a comparison. A symbol is needed to connect them. And only the whole *formula* (the children were given this term right away) tells about these results, what the weight or volume or length of one object is in comparison with the weight or volume or length of another.

In the course of several lessons, by introducing more and more new parameters (the loudness and duration of sounds, the area of figures and real objects, the strength of a blow, the composition of groups of objects to be assembled), and only a small selection of letters -- A, B, C, and D, the teacher trained the children to use the new form of transcription. In many of the problems the children were given a formula, such as $A = B$, and were to select objects which would yield this result if compared for some attribute. Here is an excerpt from the report of a lesson on September 21, 1964 (in a Moscow class, with E. S. Orlova, teacher).

Teacher: Show us the objects you have chosen. Misha, you have two new pencils there. Why did you pick pencils like those and not these (takes pencils of differing length from a pupil's desk)?

Misha V: Not those -- it says in the formula on the board that we have an equality: A is equal to B. I took pencils and compared them, and these two are equal in length (he holds them up).

Teacher: Good. What do the letters A and B tell you?

Misha V: They say that the pencils are equal.

Teacher: Is that what the letters say? They themselves, A and B here, tell about equality?

The pupils raise their hands. The class is animated.

Teacher: We won't help him for the time being! Now think.

Misha V: (after a short pause) The letters tell me about the length of the pencils -- this one and this one.

Teacher: Is that all? If the letters tell about the length, then I'll take a pencil of this length -- this is A, and one of this length -- this is B: What I get is that A is less than B.

Pupils: You can't take those -- then you have a different formula.

Misha V: We have an equality -- there is an equal sign there. We have to take pencils of equal length and then it's right.

Teacher: Then what tells about the equality itself?

Pupils: The symbol between the letters -- the whole formula.

Teacher: Now I'm changing the symbol in my formula to read A is less than B. Can you find objects to show what this means?

The children find appropriate objects; the teacher reviews the basis for the choice: the meaning of the letters, the symbols, and the formula as a whole; we should note that the children choose objects which can be compared by various parameters, some of them even demonstrating the inequality of groups of objects according to some criterion.

A special series of problems in the form of games was introduced in order to guide the children toward the idea of a "collection" of formulas by which all possible relationships could be expressed. The teacher would use the pupils' own work to show that, despite the variations among the objects being compared for length, for instance (from pencils and strips of paper to the children's own height), and despite all the differences in length of objects designated by the same "name" (such as strips of paper), one gets either an equality or an inequality, and the latter will be either "more than" or "less than." Therefore, no matter what objects are compared, we will get either the formula $A = B$ or the formula $A \neq B$. An inequality will be specified as either $A > B$ or $A < B$. The children would relate the results of particular comparisons they made to this network of formulas they had written in their notebooks. At a special lesson, under the guidance of the teacher, they excelled at choosing objects for comparing in one way or another, and the results of the comparisons always fit one of these formulas.

During this work (which was of great interest to the children, by the way), the teacher would also require them to indicate which attribute a letter designated when they were giving the results of a comparison. This is a particularly important point, since the children were actually forced to realize that the results of comparing lengths, volumes, weights, or forces could all be conveyed by the very same formulas, but that the letters in each case would tell not about the objects themselves but about their _length_ or _strength_ or _weight_.

In our view, the rule that the "general be made concrete" was very important both for dealing with the "meaning" of a formula and for correctly linking a letter (a symbol) with its object -- the _concrete_, _particular_ _value_ of some quantity. As was mentioned earlier, we attempted to organize the instruction in this topic so that the children themselves (at the first stage, in any case) would interpret the letter as the designation for the weight, volume, length, or any other parameter of a _given_ object as compared with the weight, volume, or length of _another_ object. The letter would acquire the unique function of a _general_ _symbol_ for any concrete value of a particular parameter. Since the children were actually able to derive formulas from comparisons of objects by _any_ specific values of these parameters, and by the same token, since they needed no assistance in providing illustrations for the formulas, we have grounds for believing that they were making use of this very function of the letter.

In the concluding lessons of Topic II the teacher drilled the children on the idea that the results of any given comparison can be expressed by _one_ _and_ _only_ _one_ of the three formulas which make up the established "collection." He would usually do this by presenting "clashing" formulas for the results of one comparison. Then the children would establish by discussion the wrongness of a "dual" or "triple" transcription and select the "right" formula.

A second issue discussed at these lessons concerned a rather subtle point -- the possibility of using various letters and the limits on this variability. A number of times, the teacher would not indicate which letters were to be used in recording the results of a comparison. The pupils would select letters on their own. The teacher would write the variants on the board: $A > E$; $B > C$; $F > K$, and so on, and would discuss whether these formulas were identical. With no assistance, as a rule, the children would establish that these formulas were identical, making reference to two matters -- the symbol "more than" occurred in each, and the formulas were all talking about the same result.

At the same time the teacher would give a number of examples to show that it is better to use different letters when comparing different attributes to know during this lesson which formula refers to what attribute (even though all this would lose its meaning at the next lesson since the same letters would be used in other situations).

One other odd matter shall be mentioned. At first some children (as a rule, several in each class) would record the results of comparisons using letters of different sizes; that is, they would carry over the principle of using models of the objects as symbols. The teacher would show that this is unnecessary in a formula since the relationship is indicated by the symbol for inequality. At several lessons the children would be shown formulas whose letters differed in "size," but whose meaning was the opposite of the appearance ($A < b$, for instance). They were to select appropriate objects as illustrations, going by the symbol in doing so. The teacher would demonstrate again that the letters themselves could be any "size," and that what was important was the meaning of the formula which, with its symbol, designated the comparison of "any" objects (which became an everyday expression for the pupils).

The work in Topic II (fourteen to sixteen lessons were spent on it) is a crucial part of the entire introductory section of mathematics, since in essence it has to do with setting up a special aspect of the child's activity, the system of relationships which isolate quantities as the basis for subsequent mathematical transformations. Letter formulas, which replace a series of preliminary methods of transcription, turn these relationships into an abstraction for the first time, for the letters themselves designate any specific values of any specific quantities, while the whole formula designates any possible relationships of equality or inequality of these values. Now, by relying on the formulas, it becomes possible to study the actual properties of these relationships, turning them into a special subject for analysis.

Realizing the importance of Topics I and II to the mathematics course as a whole, we made a special check of the extent to which the children had mastered them. Below we give typical results from one such individual check made during the last week of September, 1963, in the first grade at Moscow School No. 786 (with G. G. Mikulina, teacher). The children were instructed to solve three problems which would show, on the one hand, whether they had learned the methods of comparing objects (aggregates of objects, in particular), and on the other hand, whether they understood the connection between the results they had obtained and the methods of writing them down. These problems were as follows (the individual parts of them are indicated below by Arabic numerals and letters).

> Problem I. The pupil is shown a little "house" made of one big block and two little ones (Figure 7a). There are four more big blocks and six more little ones on the table. The experimenter demonstrates that new "houses" can be made from these blocks according to the model. After this the pupil is given the problem:
>
> 1. "Sort out these blocks so that we can find out whether there are enough big and little blocks to make houses like this one." The pupil must arrange the blocks in an appropriate way (such as is shown in Figure 7).
> 2. "Are there enough blocks of both kinds to make 'houses' like this one?" The pupil must answer the question.
> 3. "What kind of blocks aren't there enough of?" (There are not enough small blocks for this particular problem.) The pupil's answer should follow from an understanding of the conditions of the problem.
> 4. "What symbol can we use to record the results of comparing the two groups of blocks?" (The group of small blocks and the group of big ones are pointed out.) The correct response is to say, "The symbol 'is less than'" and to write it down.

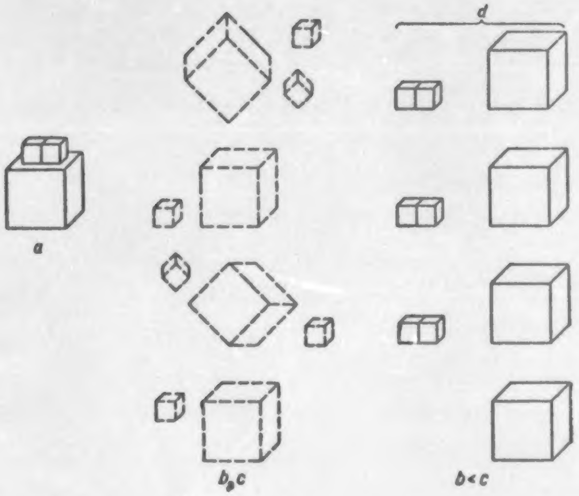

Figure 7. -- Diagrams of pupils' work in comparing aggregates of blocks: (a) is the model of the complement; (b) and (c) are parts of the complement (their original position); (d) is comparison of the parts and the results of it.

5. "Why are you using that symbol?" The pupil must substantiate his decision by citing the fact that for this particular model, there are not enough small blocks to build the houses.

Problem II.

1. Two mugs filled with water are placed in front of the pupil (the one on the left contains .5 liters, the one on the right, .25). The problem is -- "Compare the volume of water in the mugs and draw lines to show the results of the comparison." The pupil must draw two lines, the left one longer than the right one.

2. Two blocks are put in front of the pupil (a big one on the left, a small one on the right). (a) "Compare the volume of these blocks." The pupil compares the blocks. "Can the results of the comparison be shown by drawing lines?" The answer follows. (b) "But do you have to draw new lines? Or can you use the ones you already have? Why?" The pupil should indicate that he can use the lines already there for recording the results of comparing new objects.

Problem III. The pupil is given two weights (a 50-gram one on the left, a 100-gram one on the right).

1. "Compare the heaviness of these weights and record the results of the comparison in a formula. Designate this (50-gram) weight by the letter A and the other by the letter B." The pupil should write the formula $A < B$.

2. "Can the letters A and B be replaced by any other letters?" The answer follows. "Which ones? Write them down!" The pupil writes the formula using other letters.

3. "You now have the formula $A < B$ and the formula . . . (the letters may vary). Are these formulas the same or different?" The pupil should give the answer required by the sense of the problem.

4. "Why are they the same (or different)?" The pupil should substantiate his answer by making reference to the symbol and the objects involved.

Problem I presupposes the ability to juxtapose the parts of an object being assembled (a "house"), and to determine the correspondence between them (that is, to *compare* them) from the standpoint of *the requirements of the model*. The children must make an abstraction from the "particular elements" of the groups, a purely visual aspect of the situation. Problem II tests the ability to use lines to record the results of a comparison and to use the *existing* lines as the transcription of the results of a different comparison if these results are identical in their meaning. In Problem III the children's understanding of the fact that letter formulas are identical in meaning if they designate the same relationship between objects is determined.

We shall quote first from the report of one pupil's test, that of Larisa S. (on September 25, 1963), which is typical of many of the tests (all thirty-eight pupils took it). Since the experimenter's questions have been quoted in our description of the problems, they are not repeated (only their numbers are given). Only the subject's answers and reactions are given here, along with supplementary instructions from the experimenter.

Problem I.

1. She started to build "houses" but soon stopped and after a short pause took the blocks apart so that there was a big one next to each two small ones: "You can do it this way . . ."

2. "No . . . there aren't enough . . . this is an extra one." (Points to a big block.)

3. "This is an extra one -- there aren't enough little ones."

4. "Symbol?" (A pause.) "There is a symbol of inequality here!" (Experimenter: "Be more precise.") "The symbol 'is less than'" (she writes it down).

5. "This block is extra, and there aren't enough of these (small ones) for a house. There are more big ones, so you need the symbol 'is less than.'"

Problem II.

1. She draws two lines, the left one longer than the right one.

2a. "You can do it with lines" (she attempts to draw them).

b. A pause. She starts to draw new lines but stops immediately. "You can do it with these lines -- one is longer than the other, like here." (Experimenter: "Why didn't you draw them?") "You can do it this way." (She points to the lines drawn earlier.) "We have the right kind already."

Problem III.

1. She immediately writes the formula $A < B$.

2. "Be replaced? The weights?" (Experimenter: "No -- the weights are the same, but replace the letters.") "Yes, you can. I'll do it right now." (She writes $Z < P$.)

3. "The letters are different . . . The formulas are identical."

4. "The symbols are identical here and here" (she points to the formulas). (Experimenter: "How might you say that more precisely?") A pause. "The weight of the weights is written with the letters" (she points first to the first formula and then to the second one) . . . "The weights are the same -- and the formulas are identical."

This report shows that the pupil understood the questions directed to her and saw the connection between the formulas and the objects being compared. She solved the problems correctly and completely on her own.

Not every pupil responded this accurately, of course. Several needed help from the experimenter who gave either a leading question or a hint. A few pupils were not able to solve certain problems even after being given such help. In Table 1, general quantitative data about pupils' performance on the three problems, part by part are given (the number of pupils who managed to solve the problem in one way or another is indicated).

TABLE 1

PUPILS' PERFORMANCE ON THREE COMPARISON OF OBJECTS PROBLEMS: TOPICS I AND II

Manner in which the problem was solved	Problems											
	I					II			III			
	1	2	3	4	5	1	2a	2b	1	2	3	4
Solved it on their own	27	38	38	36	34	38	38	30	37	35	23	28
Solved it with the help of a leading question	5	"	"	"	2	"	"	"	1	"	4	2
Solved it with the help of a hint	6	"	"	"	"	"	"	4	"	3	5	"
Did not solve the problem	"	"	"	2	2	"	"	4	"	"	6	8

In solving part 1 of Problem I, eleven out of thirty-eight pupils needed help from the experimenter in order to sort the groups of blocks into rows. These pupils first tried to build "houses" and to compare the groups in this way. They were forbidden to do this. Only with the subsequent help of the experimenter did they classify the blocks as the problem required. In parts 2 and 3 of Problem I, the pupils performed on their own. That is, they answered immediately that there were not enough blocks -- not enough small ones, to be specific. The most crucial part of Problem I was the fourth part, which required them to recognize the relationship of inequality and to designate it by a specific symbol. Thirty-six out of thirty-eight pupils performed this task. In spite of the obvious "predominance" of the small blocks, these pupils responded as the sense of the problem demanded -- that there were fewer small blocks. Thirty-four of them substantiated their conclusion correctly (two needed a leading question).

All the pupils performed the first and second parts of Problem II. They drew lines to depict the relationship between the volumes of water and the volumes of the blocks, rapidly and with no assistance. The last part of the problem proved to be more difficult. Here they had to determine the possibility of using the lines they already had in order to describe a new result. Thirty of the pupils established this possibility on their own, and four did it after a hint. Four thought it necessary to draw new lines.

Problem III turned out to be the most difficult (especially the third and fourth parts). All thirty-eight pupils successfully replaced the letters in the preceding formula (thirty-five of them on their own; see Problem III, part 2). But only thirty-two pupils were then able to establish the identity of the formulas -- twenty-three on their own and nine with the help of leading questions and hints. Six were unable to establish the identity of the formulas. Thirty were able to substantiate their conclusion that the formulas were identical, and twenty-eight of these did it on their own (Problem III, part 4).

These data and the reports of the testing indicate that the majority of the pupils had thoroughly mastered the methods for recording the results of comparison using letter formulas, and understood the meaning of these formulas and their connection with the actual relationships between objects.[47]

Topic III -- (the properties of equality and inequality). After the children had been introduced to formulas using letters, they were ready for an explanation of the properties of relationships expressed abstractly as equality and inequality. The first of these is the reversibility of an equality (presented in the twentieth and twenty-first lessons). The teacher would demonstrate once more that the results of a comparison should be given from left to right (written

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[47] After the testing had been completed, the teacher introduced special exercises into the lessons, by which all the pupils mastered the points mentioned.

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as $A = B$, with the objects arranged correspondingly). But at this point he would reverse the objects (the red stick whose length was designated as A was now on the right, and the blue one on the left). The children would observe, first, that the objects had been transposed and second, that the results of the comparison had not been changed, that there was still an equality, written $B = A$.

Then the children and the teacher would determine again and again, by comparing various attributes of new objects, that when one reverses the position of equal objects, the external appearance of the formula changes but its meaning remains the same. The teacher would conclude: "If A is equal to B, then B is equal to A." The children would write these formulas down and draw a box around them.

Then there would be a series of exercises. For example, the children would select objects to illustrate the formula $C = E$. Then the teacher would write down the new formula $E = C$ and would ask: "What kind of new sticks (or blocks) do you need to get, to explain this formula?" They would usually answer correctly: "We don't need to get any new sticks. We already have what we need; we just have to reverse them." The children would arrange the objects to correspond to the second formula.

The following type of exercise involved "filling out" formulas with the proper letters and symbols. Formulas such as these were written on the board:

$$K = G \qquad\qquad A = B \qquad\qquad C = D$$
$$G \ldots K \qquad\qquad B \ldots K \qquad\qquad D = \ldots$$

(in place of the dots they were to put the omitted letters and symbols according to the sense of the problem; in the second pair of formulas which contains K, it was impossible to be sure what to put in, since the relationship between A and K is unknown). As the children copied down these formulas into their notebooks, they inserted the necessary elements. Many of the children would "falter" on the second one and ask whether there was a mistake in what is written, with A meant instead of K. But some children unhesitatingly put an equal sign, "spontaneously" replacing "K" with "A" in their explanation. The teacher would take this occasion to explain how to work with these formulas and then would assign a series of similar exercises.

The teacher would use the following device to explain the meaning of reflexivity to the children. He would hold up a board whose length was designated by the letter C, let us say, against the blackboard and draw a "copy" of it with chalk (the children would do analogous work in their notebooks). He would point out that the length of the line on the blackboard (or in the notebook) could also be designated by the letter C, since this was the very length from which it was obtained. If all this is written in a formula it will be $C = C$. The teacher could also say that the length of the stick is equal to itself, adding that here the stick and its "shadow" on the blackboard or in the notebook are of equal length.

Similar work was done on the area of figures as well (here, too, it was possible to make "copies"). But the teachers usually spent little time on this, since we have not been able to propose a very acceptable approach to handling this property.[48] The children made almost no errors on formal exercises, however. They would put in an "equals" sign in formulas of the type A ... A and B ... B and would put in the proper letters in such formulas as ... = A and C = ...

The next stage of the work was to explain the connection between the symbols "more than" and "less than" when the letters (or objects) in formulas of inequality are reversed. This work also helped the child to understand the meaning of the reversibility of an equality (because of the difference in the results of transposition). It was introduced in a comparable way. The positions of objects were reversed, and the new results were evaluated, written down, and compared with the old. The children did not seem to have any particular difficulty here, apparently because even when they used lines to write down the results of a comparison, in words they would often repeat the comparison in the opposite direction: "This block is smaller than that one, so that one is larger." In any case, problems on the level of "pure" formulas could be given immediately following the first demonstrations using objects. The change from the symbol " > " to

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[48] It should be mentioned that we do not think that the method indicated is the best. We are still uncertain how to explain to children the real meaning of reflexivity (and unfortunately we have not yet achieved this in the work on Topic III).

" < " and back would be accompanied by verbal formulas such as "If A is larger than B, then B is smaller than A." The children would find the most diverse objects to illustrate these "rules" and their written expression.

At this stage the pupils solved such problems as these (objects sometimes being used and detailed explanations being given):

$A > B$ $\quad$ $C = K$ $\quad$ $N < D$ $\quad$ $G < K$
$B \ldots A$ $\quad$ $K \ldots C$ $\quad$ $\ldots > \ldots$ $\quad$ $K \ldots G$

It is striking that in a number of instances the comparison of the objects would be made in one direction while the evaluation for writing it down would go in the other. For instance, some pupils would say: "This weight here is heavier than that one" (reasoning *from the right* cup of weights *to the left*). "We have to write it this way: A is less than B" (the notation going from *left to right*). It might be said that these children were "turning" the one relationship into the other instantaneously. More and more such instances occurred subsequently, so that in time this kind of reevaluation was automatic. The teachers kept calling to the children's attention, meanwhile, that the presence of one symbol, when "movement" is from left to right, indicates immediately that it is possible to move from right to left if the opposite symbol is used (this is contrasted with the immutability of equality).

In the twenty-fourth and twenty-fifth lessons the children were introduced to the *transitivity* of relationships. They worked with special sets of planks, blocks, mugs, and weights which they could set up in series of relationships from "larger" to "smaller." The children would arrange these objects in "increasing" and "decreasing" series (Figure 8). At the same time they would describe the relationships verbally (without writing them down), designating the elements of the series by letters. For instance, following the teacher's instructions and with his help, they would say: "This stick is shorter than that one, and that one is shorter than that other one"; "The length of the red stick is less than the length of this blue stick, and the length of this blue one is less than that of this white one";

"Stick A is smaller than stick B, and stick B is smaller than stick C."[49]

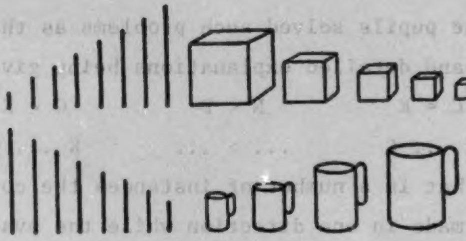

Figure 8. — Arrangement of various objects in series (by relative length and volume).

The children would also look for analogous "steps" in the objects around them. When they talked about height, the teacher would call their attention to the fact that Kolya was taller than Tanya, Tanya taller than Misha, and Misha taller than Lida (and they would make analogous multi-step comparisons by weight, hardness, composition of groups of objects, and so on).

The children also solved the following problems. With two boards (or blocks, perhaps) of differing length before them, they would first choose letters, such as A and K, to designate them. They were to select a board B such that A was larger than B but B was larger than K. These conditions were written on the board as a pair of formulas. There were similar problems in which the children were to draw lines or circles in their notebooks. There were two goals here: (a) to introduce the children to setting up "ascending" and "descending" series, and (b) to train the children to make accurate relations between the elements of formulas and the objects in the corresponding series. The latter needs special explanation.

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[49] When the children first began to use letter designations, they connected them with a particular parameter of an object such as length or weight. Thus in solving a problem they would say: "A is the weight of the object," or "the plank has length A." But gradually they would shorten these formulations, and more and more often a letter would refer to the object itself ("object B," for instance). But what was meant here, of course, was the quantities being compared. By special questions the teacher could bring the pupils back to the original detailed designations, but they would do this less and less often. In their work on Topics III and IV, they usually used only the shortened expressions.

The first variants of the curriculum included an introduction to transitivity. But no system of exercises had been worked out yet to help pupils learn to relate formulas with the objects illustrating them. The need for such became particularly apparent from special testing which showed that many children depicted the relationships indicated in the formulas $A > B$; $B > C$; $A > C$ by drawing four lines instead of three. And only a few made the two middle lines identical. The rest drew them of differing length (although the second line for B usually was smaller than the first one). To prevent such errors, special exercises are needed to help to get across the meaning of the paired formulas and the place of their middle term.

Using objects as visual aids, the children would write down the following chains of formulas, clearly distinguishing the transitional links (but they would draw no conclusion from these formulas):

$$\begin{array}{lll} A > B & K < N & A = C \\ B > C & N < M & C = E \\ C > D & M < E & E = K \end{array}$$

By relating these formulas to objects, it was possible to formulate a kind of conclusion, such as A is greater than D. The teacher would demonstrate this possibility, guiding the children toward actually determining the relationship between the extreme terms of the formulas by their connection through the middle terms. The need for using the formulas to draw a conclusion became very clear when the teacher assigned the problem: "Board A is larger than board B, and board B is larger than board C. We do not have board C here. What should it be like in comparison with A, given these conditions?" It is interesting that the children began to reason "backwards" in this case. Usually they found that "C is smaller than B and smaller than A, so that means that A is larger than C." They wrote down the formulas on the basis of this kind of verbal statement. Once they had the formulas, the statement itself became shorter and more precise and gradually turned into the standard "if . . ., and if . . ., then . . . ."

Similar problems were given in varying form for all three relationships. Some were in game form, such as having to find the hidden object on the basis of certain formulas. Find C if A and B are given and we know that $A = B$ and $B = C$. The formal notation would gradually come to look as follows (the teacher wrote the problems on the board):

$$\begin{array}{c} A > B \\ B > C \\ \hline A \ldots C \end{array} \qquad \begin{array}{c} A < B \\ \ldots. \\ \hline A < C \end{array}$$

At the start, without even needing objects as visual aids, the children (at the teacher's request) would give detailed oral explanations of each step they took. They were then told to put in only the symbols and letters needed. At various times they were asked to make a sketch to illustrate a formula they had found (they would draw lines).

In the final lessons of Topic III the teacher assigned problems requiring that transitive and intransitive relationships be *distinguished*. It was desirable for the children to "feel" this distinction, although they could scarcely be expected to provide a logical basis for it. An example of a problem is: "The boy loves the bunny, and the bunny loves carrots. Does the boy thus love carrots, too?" Or: "Tanya is friends with Masha, and Masha is friends with Valya, so that means Tanya is friends with Valya. Right?" These problems aroused a lively discussion, in the course of which the children inclined toward the view that *no* binding conclusion could be drawn. They were able to give some grounds for the conclusion. They correctly grasped, for instance, that the boy might possibly lack "love" for carrots even though he felt "love" for the bunny. The teacher would use the problems to distinguish more sharply the characteristics of transitive relationships, where the conclusion is "imperative."[50]

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[50] We should comment that further research is needed into designing a system of exercises which will make transitivity clear to the child and help him focus on it in solving problems. Our experience shows that the child comes up against some difficulties in transitivity.

The children had to solve many problems in these lessons (including word problems) without relying directly on objects as visual aids. The transition we planned to the analysis of relationships was based on certain abbreviated statements about their properties. As the children learned the material in the following topics, the possibilities for this kind of analysis kept broadening.

Topic IV -- (the operations of addition and subtraction). There were several stages in the transition to the first operations (beginning with the thirtieth and thirty-first lessons). First the teacher would simply demonstrate a change in some parameter of an object. This was most convenient to do by changing the volume of water in a flask, or the force of a push, or the weight of a load. He would also give various examples from everyday life, all of which had the same meaning: "There were so and so many, and this changed to so and so many." The children fully understood that there are two directions of change here -- increase and decrease.

The next step was to describe the change. The children compared the volume of water in two identical flasks (the water level was marked on the side), and wrote down the formula $A = B$. Then the teacher poured a certain amount of water into the left flask and proposed that the new volume be designated by the letter C and that $C > B$ be written down. But how was this new volume arrived at? Could C be obtained from the former A? How should what happened to A be written? The children indicated in some form that a certain amount of water was added to A and that C was obtained. With the teacher's help, $A + K = C$ was written and the meaning of the symbol " + " and of the letter K established. (This was "handled" by such means as going back to the former volume.)

Further, the children substituted a sum into the formula for inequality and obtained: $A + K > B$ (this point is methodologically difficult and requires certain "moves" which are not described here). The formula $A - K < B$ was obtained in a parallel manner, as well as the formulas $A < C + K$ and $A > B - K$, where problems were solved using various parameters of various objects (it was especially convenient to use the weight of dry substances).

At the next stage the children carried out operations with visual aids according to formulas the teacher indicated. Given the formulas $C = D$, $C < D \ldots$, for instance, they determined the direction of change and reproduced it using strips of paper. As they did this, they reasoned out loud: "They are equal, and if the left side becomes smaller, that means that the right side is larger, that it has increased."

A few lessons later the children were given a new problem: What had to be done to make the sides equal again? In every class nearly a third of the children gave an immediate answer: "You have to subtract" (if something had been added) or "You have to add" (if something had been subtracted). With the teacher's help they checked this method, and it was correct. But hardly anyone was able to find the other method -- that of changing the other side of the inequality. Although the teacher would show that this was also possible, by now a substantial number of the children (about half of each class) could determine the amount of the change -- "by the same amount" -- on their own. The formula of the type: $A + K = B + K$ was added to the previous formula for inequality.

However, in some classes two or three pupils noticed a discrepancy in the way the formula was written and proposed writing it this way: $A + K = B + G$ where $K = G$. That is, the second items themselves should be represented by different letters. If the pupils did not come upon this way of writing it themselves, the teacher would show it to them, and then it would be used along with the first way.

As the pupils performed various exercises, they became more and more skilled at explaining their methods of operation verbally, and with each lesson they had less need to use objects as aids. To solve problems they would "mull over" (aloud or in a whisper) the possible relationships in the given conditions. Problems were written as follows:

$$\begin{array}{lll} A = B & A = D & B = C \\ A + D \ldots B & A \ldots < D & B > C \ldots \\ & & B \ldots = C \ldots \end{array}$$

The children's reasoning constantly revealed that they understood the real connection among the various types of relationships:

"A was equal to B. B was decreased, it became less than A, so that means A became larger -- we have to put in the symbol 'more than'" (Andrei L. from Moscow); "The right side of the equality was increased, so that means we have to put the symbol 'less than' -- if A equals B, then A is less than B increased by C" (Vova S. from Tula).

By introducing the children to the operations of addition and subtraction, it becomes possible to broaden the range of "word" problems whose conditions contain letters as data.

Since the next topic is different, the results of the first four topics are summarized, particularly the results of work with the letter formulas. A series of individual tests using specially selected problems on the topics which had been covered was given in one Moscow class (G. G. Mikulina, teacher) during 1963-64. For the children the tests were "unexpected" in a number of cases, since they included problems on which they were not working directly at the time and which they might have "forgotten." The tests made it possible to explore how much they had learned and how well.

Data concerning performance on these tests, which included problems of three types, are given in Table 2. The eight problems of type I called for the ability to write appropriate letter formulas after observing certain relationships between objects ($A = B$; $A < B$; $A > B$). The seven problems of type II required a knowledge of the basic properties of equality and inequality (insertion of symbols in formulas of the type: $A = B$, $B \ldots A$; $C > D$, $D \ldots C$; $K \quad N$, $\ldots > \ldots$, and so on). Finally, the eighteen problems of type III dealt with all the ways of going from equality to inequality, and with returning to equality through addition and subtraction (such formulas as: $A = B$, $A + K \ldots B$, $A + K \ldots B + K$; $C = G$, $C \ldots G - D$). The problems of types II and III were to be solved only through analysis of the letter formulas, with no dependence on objects as visual aids. (The tests were given from September 30 to November 1, 1963.)

The majority of the pupils made no errors in solving the problems of types I and II. The number of errors increased in the problems of type III, which the pupils were to solve rapidly and "to themselves." Almost all the errors on the November 1 test occurred with "difficult" formulas: $A = C$, $A < \ldots$; $A = D$, $A + K \ldots D + K$. Although twenty-

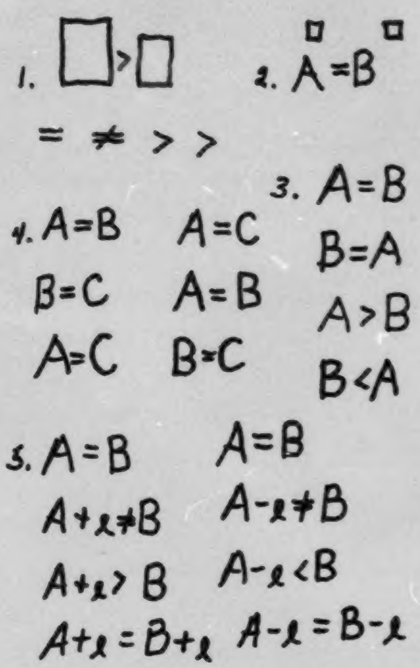
TABLE 2

FREQUENCY OF ERRORS ON THREE PROBLEM TYPES: TOPICS I - IV

No. of tests with --	Problem type[a]						
	I	II		III			
	8:Oct. 1	2: Sept. 30	5:Oct. 2	5:Oct. 8	3:Oct. 23	5:Oct. 28	5:Nov. 11
No Errors	31	34	29	36	31	28	26
One Error	3	1	5	1	5	9	3
Two Errors	1	..	1	1	..	1	7
Three Errors	1	..	..	..	..	..	1
Possible Errors	288	70	175	190	108	190	185
Actual Errors	8	1	7	3	5	11	20
Percent Errors	2.8	1.4	4.0	1.6	4.6	6.0	10.0
No. of Subj.	36	35	35	38	36	38	37

[a] 8:Oct. 1 means 8 problems, test administration on Oct. 1

six pupils out of thirty-seven made no errors at all on this test, it obviously caused the children a certain amount of difficulty. At the same time the summary data of performance on the problems of type III (out of 673 possible errors 39, or 5.8%, were committed) show that many of the children had a firm knowledge of the material in Topics I - IV (Figure 9).



1. $\square > \square$

2. $\overset{\square}{A} = \overset{\square}{B}$

$= \quad \neq \quad > \quad >$

3. $A=B$
$B=A$
$A>B$
$B<A$

4. $A=B$ $\quad A=C$
$B=C$ $\quad A=B$
$A=C$ $\quad B=C$

5. $A=B$ $\quad A=B$
$A+\ell \neq B$ $\quad A-\ell \neq B$
$A+\ell > B$ $\quad A-\ell < B$
$A+\ell = B+\ell$ $\quad A-\ell = B-\ell$

Figure 9. -- Results on individual tests, over Topics II - IV, taken by Moscow first-grade pupil Tanya K: (1) representation of the results of comparison using symbols; (2) the first letter formula; (3) the properties of equality and inequality; (4) the transitivity of equality; (5) the violation of equality and its "restoration" through addition and subtraction (she performed items 3 - 5 in her "head" without using objects to aid her).

Topic V -- (the shift from inequality to equality).[51] Once the children had made a comparison and had recorded the results of it with the formula $A < B$, they were given the new challenge of turning this inequality into an equality. With an object before them as a visual aid, many of the children would indicate on their own how this might be done -- they would propose making B smaller or A larger. By working with boards or strips of paper they even were able to demonstrate this method. Then, at the first opportunity (the fortieth or forty-first lesson of the year) the teacher gave the notation for this information: $A < B$, $A + \ldots = B$, $A = B - \ldots$.

The dots conveyed to the children that something was added or subtracted with the result being an equality. By demonstrating with flasks of water or with weights, the teacher showed that it was not known beforehand how much needed to be added or subtracted (relatively long pieces of string could also be used). "Something" needed to be added to A, and the "needed" part could even be written in advance, but what this "something" was exactly and "how much," was not yet known.

Together with the children the teacher would establish that by writing such a formula down, they were only contemplating or planning an "increase" or a "decrease." The teacher would propose the special symbol x to designate the unknown in this formula; that is, that which needed to be indicated in order to reduce the inequality to an equality: $A + x = B$; $A = B - x$.

As a rule, the children rapidly grasped the meaning of this symbol. Thus the very first lesson many children were able to correctly explain that not just "any" weight, for instance, or volume could be added or subtracted, but that the difference between A and B needed to be known and that it was not yet known.

Several subsequent lessons were spent introducing the children to methods of determining this "difference," with objects used as visual aids. It was important here not simply to show ways of working with the objects (such as laying boards together or pouring water into flasks), but also to teach the children to use letters to describe these processes and their results.

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[51] In an article [11] A. A. Kiryushkina has described the main features of the work done in this topic, as well as typical difficulties which the children have in solving elementary equations. Therefore, we shall only take time here to give the general results of the work done in this topic.

This was the most difficult part of the whole topic (see the article by Kiryushkina [11]). The child encountered a new meaning of subtraction when he wrote $x = B - A$. It was not an actual diminution (as it was in Topic IV), but only a formal description of the comparison of quantities B and A, where B as an object remained the same as it was and the quantity corresponding to x was to be obtained from other material. This notation is only a formal description of the process of obtaining x. Since the children had actually determined x (the "difference"), they would then "add" it to A and obtain the required equality. The majority of the children understood the meaning of the letter description of "addition" right away, although many of them were still confused by the way it was written since it had several letters and parentheses in it. In sum, the entire process of making up and solving an equation with the aid of objects can be seen in the following system of formulas:

$A < B$ (the initial condition);

$A + x = B$ (the planned transformation);

$x = B - A$ (the search for the "difference");

$A + (B - A) = B$ (the actual equating).

The children would write these down under the teacher's supervision. Then exercises were gradually introduced in which the children were either to observe a fellow student working with objects or to do the work themselves, and then were to use formulas on their own to describe the entire sequence of the conversion of an inequality into an equality. The teachers encountered certain difficulties here, resulting mainly from the children's lack of skill in organizing their own work. But after several lessons nearly all (with the exception of three or four who still needed direct assistance) were able to solve these problems with only minor errors.

The next stage consisted of the gradual transition to solving equations on the level of the formulas themselves. The way was prepared for this transition by extensive work with word problems. The children would be given a problem such as: "There are A kilograms of apples in one box and B kilograms in another. We know that A is less than B. What needs to be done so the apples in the first box weigh the same as those in the second?" The pupils would quickly write down the condition and with no mistakes, as a rule, would propose a way of solving it. Some apples need to be added to the first box. Writing the equation down caused no difficulty either: $A < B$, $A + x = B$. The children understood clearly that they now had to find x.

At this point the teachers would usually use graphic means in the search for x. The children would draw lines to depict the weight of the apples. Then there would be a discussion of the way to find the "difference" in weight (actually the teacher had scales on his desk and he would imitate the search for the weight of the apples). Line A would be "superimposed" on line B, and the remainder, expressed as $B - A$, would be defined as being equal to x. Using this as a model, the children, with the teacher's help, would do the necessary weighing and find the weight equal to x. Then this weight (or line segment, correspondingly) would be added to A and the final formula written. Other problems were solved similarly. The goal was for the work with formulas, which was first accompanied by simultaneous operations with objects, to be combined with graphic representation, and gradually to become relatively independent both in its meaning and in the order in which it was done.

This direct transition was preceded by work involving an "intermediate" formula which performed a function of its own. By the time they had written the equation down, the children would usually have already "unplugged" the initial inequality (in fact, sometimes they would have received the equation itself "second hand"). It was necessary, therefore, to return to the initial formula, but this time from the equation: If $C + x = D$, then $C < D$. Or rather, it was necessary, through this "repetition" of the formula, to make the connection, as it were, between the equation and the inequality in order to go from it to subtracting "the less" from "the more." The whole process took on the following appearance:

$A > B$,

$A = B + x$,

$A > B$ (this formula is given "in the margin," so to speak), so then

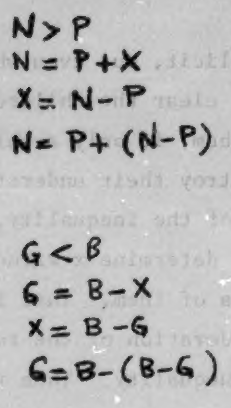
$x = A - B$.

After a while the "repetition" became implicit, but even when the first formula of inequality was completely clear the children needed to have the meaning of it repeated to them, if only a whisper. Shifting to the equation seemed somehow to destroy their understanding of the initial relationship between the parts of the inequality.

Gradually the children came to be able to determine x without needing to rely on objects or graphic analogues of them. That is, they could determine it through a theoretical consideration of the relationship between the sides (or the parts) of the inequality. Then they would substitute the values they had found into the equation. The parentheses here helped the children to understand the difference as a kind of unit of an actual quantity. (We should mention that the terms equation and difference were being used regularly.)

The work on Topic V required a rather long time -- twenty-five to thirty lessons. But the children were practicing many skills during this time, having to do with understanding the properties of the relationships of quantities to which they had been introduced in the preceding topics, and they were perfecting the "techniques" of working with complex formulas (Figures 10 and 11 show pages from tests taken by first-graders from Moscow and Torzhok, involving setting up and solving elementary equations).

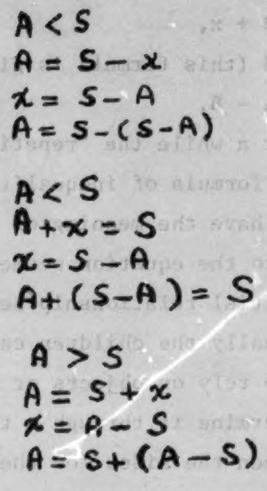
The results of the work on this complicated topic can best be seen in the way pupils performed on special tests and problems they did on their own. Table 3 gives the relevant data for a first-grade class in Moscow (G. G. Mikulina, teacher), indicating the results on problems having to do with the following basic stages of work: making up equations (problems of the type: $A > B$, $A = B + x$), solving equations ($x = A - B$), substituting the value of x ($A = B + [A - B]$), moving from equality to inequality ($A + x = B$, $A < B$). The texts of the tests are not being quoted since the problems on them are simply variants of the ones shown above. The table contains data



$N > P$
$N = P + X$
$X = N - P$
$N = P + (N - P)$

$G < B$
$G = B - X$
$X = B - G$
$G = B - (B - G)$

Figure 10. -- Test (Dec. 10, 1963) taken by Moscow first-grade pupil Misha N.



$A < S$
$A = S - x$
$x = S - A$
$A = S - (S - A)$

$A < S$
$A + x = S$
$x = S - A$
$A + (S - A) = S$

$A > S$
$A = S + x$
$x = A - S$
$A = S + (A - S)$

Figure 11. -- Test (Jan. 26, 1963) taken by Vova. E., a first-grader at Torzhok.

for only certain tests -- only letter formulas were used in solving all of these problems. (It is helpful to indicate the dates on which the children first began to do work of this sort on their own. They began to make up equations on October 21, 1963, to solve them on November 10, to substitute on November 16, and to find inequality on November 4.)

From the data in Table 4, it is evident that a substantial number of the children were able to make up equations without mistakes from the time they began work on Topic V. In the concluding lessons there were very few mistakes (4%), even on the especially difficult tests (such as the one on December 10). Solving the equations and determining the value of x also proceeded satisfactorily from the very beginning (7% error on the first test, and 4.6% on the final one). Nearly all children (35 out of 38) were able to find x without any mistakes (see the December 10 test).

On the whole the children learned how to move to inequality (although a small number of children made persistent mistakes). But substituting the value of x into the formula for the equation caused the greatest difficulty. Only ten out of thirty-nine pupils solved the one substitution problem they were given on the November 20 test (that is, four lessons after they had begun to work on it). The performance improved in subsequent days, but even so almost half the pupils were unable to learn how to do this substitution. This difficulty seems strange at first glance, since on the surface substitution appears to be "mechanical" work. But the explanation became much clearer when the errors children made were analyzed.

On the December 10 test, twenty-two pupils out of thirty-eight solved all four substitutions correctly, seven pupils solved only three (that is, made one error), six pupils solved two (that is, made two errors), and three solved one (and made errors on three). Many of the errors followed a definite pattern, however. All the children could determine x in this (or an analogous) problem: $A < C$, $A + x = C$, $x = C - A$. But instead of substituting it into the appropriate place in the equation $(A + (C - A) = C)$ they would write: $C - (C - A) = A$, or $A = C - (C - A)$. There were no errors here from the "technical" standpoint but the children had not clearly understood the real meaning of substitution. That is, the replacement of an unknown with a known. They used this "known" in order to obtain a new equality right away, without x.

Analysis of these and similar errors indicates the need for special work to familiarize children with certain formal aspects of working with mathematical symbols. In general, however, the data cited in Table 3 supplies evidence for assuming that in principle the material in Topic V is understandable to first-graders and that they are capable of learning it.[52]

In some of the experimental classes (G. G. Mikulina's class, in particular) the work on equations was continued. The children were

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[52] During 1962-63 T. B. Pustynskaya used our curriculum to teach one first-grade class at Torzhok, and among other things she skillfully explained to the children certain formal aspects of substituting the value of x in letter formulas and the meaning of this operation. Testing indicated that even with the most varied problem requiring the substitution of the value of x, the number of mistakes was very small (only two or three pupils made any).

TABLE 3

FREQUENCY OF ERRORS ON PROBLEM TYPES: TOPIC V

No. of tests with --	Problem type**								
	Making up Equations					Solving Equations			
	3:Oct. 23	6:Nov. 1*	8:Nov. 13	4:Nov. 26	4:Dec.10*	6:Nov. 13	2:Nov. 19	5:Nov. 30	4:Dec. 10*
No Errors	23	18	31	38	32	28	33	36	35
One Error	12	10	4	..	6	7	2	1	1
Two Errors	1	4	1	1	..	1	2	1	1
Three Errors	..	4	1	..	..	1	1	1	..
Four Errors	..	1	1	..	..	1	1	..	1
Possible Errors	108	222	304	156	152	228	78	195	152
Actual Errors	14	34	13	2	6	16	13	6	7
Percent Errors	13.0	13.5	4.3	1.3	4.0	7.0	16.7	3.1	4.6
No. of Subj.	36	37	38	39	38	38	39	39	38

TABLE 3 (Continued)

FREQUENCY OF ERRORS ON PROBLEM TYPES: TOPIC V

No. of tests with --	Problem type							
	Substituting				Moving to Inequality			
	1:Nov. 20	2:Nov. 22	2:Nov. 23	4:Dec. 16	5:Nov. 4	2:Nov. 7	4:Nov. 19	3:Nov. 21
No Errors	10	24	18	22*	23	30	28	30
One Error	29	10	12	7	14	1	6	7
Two Errors	..	4	8	6	1	6	3	..
Three Errors	..	..	..	3	..	..	..	..
Four Errors	..	..	..	..	..	..	..	..
Possible Errors	39	76	76	152	190	74	148	111
Actual Errors	29	18	28	28	16	13	12	7
Percent Errors	74.4	23.7	36.8	18.4	8.4	17.6	8.1	6.3
No. of Subj.	39	38	38	38	38	37	37	37

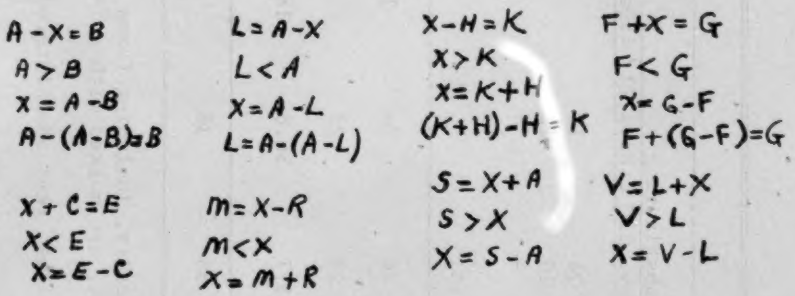
tests of greater difficulty, including forms of problems which the children did not expect, are marked with a "".

**3:Oct. 23 means 3 problems, test administered October 23.

introduced to equations of the type: $x - A = B$ and $x + A = C$ (their solution was based on the following statements: If $x - A = B$, then $x > B$ by A. Thus $x = B + A$).

In the course of this work the teacher would focus on having the children structure detailed statements, and train them not to be afraid of making mistakes (at first). The meaning of each formula (with varying letters) was discussed and "weighed" carefully and unhurriedly. On a test on December 27, 1963, the children in G. G. Mikulina's class were given these four formulas, along with some others: $x + C = E$, $x - H = K$, $S = x + A$, and $M = x - R$ (they knew certain letters of the Latin alphabet by this time).

The method of solving these and several other equations is shown in Figure 12. The results are as follows: Twenty-six pupils out of thirty-nine made no errors in solving the four equations mentioned; eight pupils missed one equation, two missed two equations, two others missed three, and just one missed them all. The total number of correct answers possible was 156, and 22 of them, or 14%, were missed. We believe that these results are not bad at all for such difficult problems.



$A - X = B$
$A > B$
$X = A - B$
$A - (A - B) = B$

$L = A - X$
$L < A$
$X = A - L$
$L = A - (A - L)$

$X - H = K$
$X > K$
$X = K + H$
$(K + H) - H = K$

$F + X = G$
$F < G$
$X = G - F$
$F + (G - F) = G$

$X + C = E$
$X < E$
$X = E - C$

$m = X - R$
$m < X$
$X = m + R$

$S = X + A$
$S > X$
$X = S - A$

$V = L + X$
$V > L$
$X = V - L$

Figure 12. -- Test (Dec. 27, 1963) taken by Moscow first-grader Masha S.

Understandably, the following question may be asked: Just how worthwhile is it to work on elementary equations and to solve them in this form? Usually, of course, they are solved by transferring letters from one part of the equation to the other and changing the sign to its opposite (the further issue of negative quantities comes up here as well). It certainly is not maintained that the approach spelled out above is the only one possible or that its value has been completely proven from the methodological standpoint, a matter which needs further discussion and testing. Another matter, however, is particularly important to emphasize. From a psychological standpoint, instruction by the experimental curriculum has revealed potentialities in the seven-year-old child for analyzing rather abstract relationships which traditional child psychology has never clearly noted. In their work with the simplest letter formulas, the children showed that they have a lively taste for *reasoning*, making *mental comparisons*, and giving a *logical appraisal* of various relationships. The designers of academic subjects face the task of satisfying this awakened interest the primary schoolchildren have shown.

At the same time, an introduction to equations written with letters is important to the development of the first-grader's skills at making *mathematical models* and *describing* actual physical quantities and the changes in them. This, our experience shows, is quite essential to all subsequent mathematics instruction, especially the solution of so-called word problems which have letters as data.[53]

Topic VI -- (addition and subtraction of equalities and inequalities: substitutions). In this topic much of the information the children had acquired earlier about the properties of relationships was synthesized. In assigning the children problems dealing with the addition or subtraction of equalities or inequalities, the teacher did not try to give the children formal rules which, after all, are provided in the systematic school algebra course. What was important was to inculcate in the pupils the ability to use *elementary reasoning*, based on the properties of relationships, and the ability to approach elementary formulas from the standpoint of their *meaning* rather than of a superficial combination of some of their characteristics.

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[53] As noted above (see p. 130), one of the most important properties of quantities manifests itself when even the simplest equation is being set up: For any $a > b$ there is always a definite quantity c such that $b + c = a$ (see Kolmogorov's axioms [12:340]).

Thus, in working on Topic VI, the children solved problems of the following type (there were many specific variants here):

$$\begin{array}{ccc} A = B & B > C & E < B \\ K > M & N = D & M < G \\ \hline A - K \ldots B - M & B \pm N \ldots C \pm D & E + M \ldots B + G \end{array}$$

After performing a number of exercises designed to acquaint them with the form of the problems, the children solved these problems quite successfully on the whole (they showed particular interest in them since they required work not yet "sanctioned" by the rules).

Finally, a certain amount of time in Topic VI was spent showing the pupils how to replace some value of a quantity by the sum of two, three, or more items ($B = C$; $C = A + D + K$; $B = A + D + K$). In a series of special exercises the children would "expand" or "contract" letter formulas according to the operations indicated (for instance, they were to rewrite the inequality $A > B$, given the condition that $A = K + M + N$). All this served as good preparation for the subsequent introduction to the commutative and associative properties of addition.

The basic stages of the first semester's work has been outlined according to the curriculum we devised, as well as the extent to which children have succeeded in learning it. Since we believe that the curricular material is important for later progress in elementary mathematics, it is logical to ask how long children retain this knowledge (if they do not, then they cannot build on it subsequently).

An answer to this question is provided by the results of special tests administered at the end of the year and at the beginning of the second grade.[54] On May 28, 1962, for instance, the following test, consisting of twelve problems covering much of what had been studied during the year, was given to a first-grade class at Tula (M. A. Bol'shakov, teacher):

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[54] In another article [3] we have given data about the performance of three classes of first-graders (in Moscow, Tula, and Mednoe) on a complicated series of test problems given at the end of February, 1962.

1. $A < B$
$A \ldots = B$

2. $A > B$
$A \ldots = B$

3. $M = D$
$M \ldots > D$

4. $M = D$
$M \ldots D + G$

5. $A - x = B$
$A \ldots B$
$x = \ldots$

6. $A + x = M$
$x = \ldots$

7. $A = B - x$
$A \ldots B$
$x = \ldots$

8. $A = B$
$A + E \ldots A - E$

9. $A < C$
$A - K \ldots C$

10. $M + E = D + E$
$M \ldots D$

11. $A > B + K$
$A \ldots B$

12. $K = M$
$K - A \ldots M + A$

The results achieved by the thirty-four pupils are presented in Table 4. The most difficult problem turned out to be No. 6, in which the children were to find x without being given the relationship between the known quantities in expanded form beforehand (nine out of the thirty-four pupils missed it). Because problems 5 and 7 were written in expanded form and were solved with more success, it may be assumed that some of the children had not learned to mentally evaluate the relationship between quantities within an equation (it was noted on page 190, the role this ability plays).

TABLE 4

NUMBER OF PUPILS (OUT OF 34) MAKING NO ERRORS ON EACH PROBLEM

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Number of pupils	32	34	33	34	30	25	31	34	32	31	32	31

Many of the problems were solved correctly by the majority of the pupils. Only twenty-nine (or 7%) were missed out of the total 400. Twenty of the pupils made no errors on any problems; eight missed only one; three missed two or three, and three others missed four or five of the twelve problems. The general results on this test show that the majority of the pupils did successful work during the first semester and learned the material quite well.

When the instructional results were outlined, we went into detail about the work done in G. G. Mikulina's first-grade class during 1963-64. To give a more complete picture it is useful to summarize the results on a complicated test which the pupils of that class took at the very beginning of their second year (on September 12, 1964). The following problems were given:

$b - c > k + m$
1. $\ldots = \ldots$
2. $\ldots = \ldots$

$a - b + c = d - m$
3. $\ldots < \ldots$
4. $\ldots = \ldots$

$a + m = b$
$d < k$
5. $(a + m) - d \ldots b - k$

$5 + 2 = 7$
6. $5 + 2 - a = 7 \ldots$

$10 = 10$
7. $10 - 2 - 2 = 10 \ldots$

$20 = 20$
8. $20 \ldots = 20 + b$

On the surface these problems appear different from the ones the children had been working on in the first grade (they had not worked with such "complicated" formulas before). Problems 6 - 8 required an understanding of the basic properties of quantities presented as numbers. The results achieved on this test by the thirty-seven pupils are given in Table 5.

TABLE 5

NUMBER OF PUPILS (OUT OF 37) MAKING NO ERRORS ON EACH PROBLEM

Problem	1	2	3	4	5	6	7	8
Number of pupils	35	32	34	29	29	37	35	37

The majority of the pupils were able to solve problems 1 and 2, in which they needed to know two methods of arriving at equations.

Nearly all solved Problem 3 as well (it necessitated increasing the right side or decreasing the left side of an equality). The weakest performance was on Problems 4 and 5. In the one they had to preserve an equality by increasing or decreasing both sides the same amount and in the other, they had to grasp the meaning of a new formula. Nearly all the children were able to solve the problems involving the numbers. Twenty out of the thirty-seven pupils solved all the problems with no errors; nine pupils missed only one problem; six others missed two, and two missed three or four. An example of the correct solution of Problems 1 - 4 follows (done by Tanya V.).

$b - c > k + m$ $\qquad$ $a - b + c = d - m$

1. $b - c = (k + m) + x$ $\qquad$ 3. $(a - b + c) - z < d - m$

2. $(b - c) - x = k + m$ $\qquad$ 4. $(a - b + c) - z = (d - m) - z$

Out of 296 possible errors there were 28, or less than 10%. If we take into consideration that this test was given unannounced, after the summer vacation, and that it included complicated formulas, we may say that in general the results on it are satisfactory. They show that many children in the first grade had gained a thorough understanding of the fundamentals of moving from equality to inequality and back and could apply these when working with either letters or numbers. By following the experimental curriculum the teachers were thus developing in the children a sound knowledge of the basic properties of quantities and the operations on them.

What are the prospects for making use of this particular knowledge? They are threefold. First, once children understand the basic characteristics of equality and inequality and the ways of moving from one to the other, their work with numbers can be focused not only on the "pure" technique of calculations but also on the structural relationships which regulate these calculations. In particular, it becomes possible for them to see more clearly the unity of addition and subtraction (and subsequently, that of multiplication and division) and how a change in the result of operations depends on a change in the component parts. In other words, there is another, more fruitful approach to the study of operations on numbers than that found in traditional teaching.

Second, the work with quantities serves as a suitable basis for an introduction to numbers, both whole numbers and fractions (see above). By putting emphasis on the *properties of quantities*, it is possible to decrease the gap between whole numbers and fractions, a very important step which needs to be taken in structuring elementary mathematics as an academic subject.[55]

Third, from the very beginning working with quantities and abstracting their properties has to do with *letter symbols*, through which the child can begin to examine particular *relationships* among objects, a matter of no small importance for all subsequent progress in the academic subject of mathematics.

In our view, the points enumerated are justification enough for including a special *prenumerical section* in the elementary mathematics course, and they suggest the merits of such a section. By acquainting the child with the basic properties of quantities, this section lays the foundation for the subsequent detailed introduction of whole numbers and then for a "smooth" transition to real numbers, and it makes it possible to "soften" the sharp opposition teaching traditionally sets up between these types of numbers, and thereby to algebraize the regular elementary school mathematics course.

The materials cited above show that *there is nothing about the intellectual capabilities of primary schoolchildren to hinder the algebraization of elementary mathematics*. In fact, such an approach helps to bring out and to increase these very capabilities children have for learning mathematics.

Data has been gathered which describes individual differences in pupils' response to the experimental curriculum. Since we will not have an opportunity to spell them out in this book, we shall mention only that even with these differences, the majority of the pupils in each class mastered the curricular material quite satisfactorily (with many getting A's or B's). At the same time there were two or three pupils in each class who needed supplementary work and who had difficulty with parts of the curriculum (many teachers face a comparable situation when they teach by the regular curriculum). We should also mention that the work described above did not overload the pupils at all, since it was aimed at finding and developing their intellectual potential rather than at increasing the difficulty of the material in a purely mechanical way, something we consider unacceptable in experimental work of our type.

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[55] In one experimental third-grade class in Moscow, fractions were introduced at the end of 1964-65, based on the measurement of quantities. The pupils were successful in learning this material, and there is reason to believe that it could be presented at the beginning of the third grade or even at the end of the second. This research, which was carried out in our laboratory by the Yugoslav psychologist Cvetkovič, shed some light on the psychology of introducing children to quantities and whole numbers as viewed from the standpoint of the subsequent introduction of fractions.

All of this permits us to assert with some assurance that there is an _inherent connection_ between the material to be learned and the intellectual capabilities for learning it. The key problem in setting up an academic subject, in fact, appears to consist in groping for this connection and _providing teaching material_ which will bring out, consolidate and develop intellectual capabilities (these "capabilities" themselves are gradually transformed into mental "ability," in the broad sense of the word).

In our experimental research we have isolated the following (but not the only) characteristic features of an academic subject which succeeds in doing this. A large part of it is given over to _introducing_ the child to the realm of material objects which will serve as the source of the relevant concepts. The child has to learn how to _operate_ in this realm before he can make the transition to full-fledged concepts. Special analysis is needed to determine the range of properties of the objects, as well as the operations the child needs to learn. For instance, by solving special problems of matching and assembling and then of comparing, the child learns how to isolate specific relationships among objects which can be converted into a quantity.

An important role in this process of isolating relationships belongs to _intermediate means_ of depicting and describing the results of operations on objects. Fully formed concepts sometimes show no trace of these means, which are significant in that they make it possible to model the properties of objects in the form of an operation to be performed. When this operation is eliminated, there is no further need for these intermediate means of description. Now it is as if the concept itself and the symbolic means of expressing it refer directly to the properties of the object. Intermediate means of description are of decisive importance to the academic subject since they serve as the intermediary between an object and some property of it reflected in a concept. Our experimental curriculum was successful because just such means (as "copied" and "abstract" sketches to help isolate relationships involving comparisons) happened to be discovered and made a part of the teaching process.

This academic subject possesses still another characteristic feature. By teaching the child to work with visual aids it gives him an understanding of the general features an object has which may be seen in these aids and may be studied later. These general features, as it were, are an indicator of the specific form new knowledge will subsequently take. Thus, when we delineated the field of scalar quantities we thereby outlined in prospect a whole cluster of specific mathematical disciplines grouped around the concept of real number.

Succeeding in making the particular visible through the general is a characteristic feature of the kind of academic subject which awakens and develops the child's ability to think theoretically at the very time when he is studying it.

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DEVELOPING THE CONCEPT OF NUMBER BY MEANS OF THE RELATIONSHIP OF QUANTITIES*

G. I. Minskaya

The concept of positive whole numbers is basic to the entire study of arithmetic in the primary grades. With a grasp of this concept and of certain properties of the decimal system, children can learn how to add, to subtract, to multiply, and to divide numbers in the course of four years. Ways of introducing the concept of whole number and counting in the first grade have been worked out in great detail in the methodology of teaching arithmetic. There is an extensive bibliography on the psychology of developing the initial concept of whole number and elementary counting skills. It would appear that this part of the curriculum and the methods involved have been firmly established. For several decades they have remained essentially unchanged. Examples are found in the first-grade textbook by A. S. Pchelko and G. B. Polyak [8], and in the corresponding methods manual by Pchelko [7]. Methodological research has been focused mainly on improving particular ways of presenting the established curricular material.

But recent psychological studies, both here and abroad, have cast doubts on the accepted content of the initial sections of the arithmetic course and have outlined new approaches to introducing the concept of number into the course. Certain studies (such as those by P. Ya. Gal'perin and L. S. Georgiev [5], and by V. V. Davydov

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*From Learning Capacity and Age Level: Primary Grades, edited by D. B. El'konin and V. V. Davydov, Moscow, Prosveshchenie, 1966, pp. 190-235. Translated by Anne Bigelow.

[1]), have concentrated on children who were taught counting and number using the customary curriculum and methods (in kindergarten and in school) and have sought to determine what attribute these children focus on when counting a series of objects. It has been found that for many children this attribute is what makes an object distinct in space and time from the others in the particular aggregate. The children, even though they were quite capable of counting out separate objects and had a clear "notion" of each number (up to ten or fifteen, usually), either were completely incapable of counting at all or else made gross errors as soon as a problem required them to count out objects on some other basis than by the separate elements of an aggregate.

When the "breakdown" of this previously established operation was analyzed psychologically, it was found that the children identify a set of units, such as the elements of a series of numerals, with the parts of a very real aggregrate. The children make no distinction between what is being counted and the particular means by which the results are represented, that is, the standard set of separate units. They identify the unit with the separate elements of the set being counted. Thus, if they are given a set of blocks and asked how many there are, their only answer will be "six," because they mentally "narrow down" the question, interpreting it on the basis of the visibly separate blocks given, and find that there are "six" such "units."

It can readily be seen that a child will "narrow down" the question and respond this way only if he already identifies the unit (the numeral "one") with a separate element (the block) of the aggregate. Here the numeral becomes just a new name for this separate object.

In principle, however, a collection of blocks can in itself be defined by various numbers, depending on the base used for counting (the measure that is selected), which may or may not coincide with the "individual" block. In Figure 1, if "the rectangle" is selected as the base then the collection of blocks is defined by the number "1." It may be also defined by the number "2" (the number of horizontal rows),

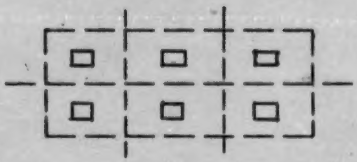

Figure 1. -- The possibility of using various numbers to describe one group of objects (depending on the base for counting).

or the number "3" (the number of *vertical rows*). The collection is defined here as a certain aggregate of units (1, 2, 3, or 6). These units designate the *relationship* between what is counted and *what* base (or measure) was established in advance and, taken separately, are a special kind of *model* of the relationship. But the units do not merge with the actual, physically distinct objects of the thing being counted which is the reason any *collection* of elements can be designated as the *individual unit* (to be called "one") if a "fractional" measure is taken as the base for counting.

When the mental operation of counting (see [2]) is fully developed, the person needs no special detailed instructions to be able to distinguish, *by himself*, what base for counting is needed ("needed" according to the conditions of the *practical* problem), to use this base, and to find the relationship between it and what is being counted without any particular conscious effort. If he can *change the base* for counting rapidly and freely, and can *keep in mind the interrelations* among the object, the measure, and the number, he has a grasp of the actual *form* of number as a special means for modeling the relationships among concrete physical objects.

Unfortunately, as we mentioned earlier, there are many first-graders who do not have a grasp of this form. The fault lies, to a great extent, with the accepted arithmetic curriculum and teaching methods, which do not take into account the actual *psychological* mechanism of counting as a mental operation or the conditions in which it can develop fully. With these methods, children do not learn to distinguish *what* is being counted, the *base* for counting, and the *means for representing* the relationship between them. Thus their notion of counting is defective, since it lacks precise points of reference for a flexible change in the base as well as for understanding that the number obtained depends on this changeable base for counting.

It is well known that according to the usual methodology, learning how to count (up to ten) includes:

(1) knowing the names of the first ten numbers and their order;

(2) understanding that when one counts an aggregate, the last numeral named tells *how many* objects there are altogether in that aggregate;

(3) knowing the place of each number in the natural series; and

(4) having a notion of the magnitude of the aggregate designated by a number [7:143].

Let us examine the most striking items in this list. In item 2, the child must understand that the numeral he has obtained designates the number of objects in the particular aggregate. In item 4 this fact is further emphasized. The child must have a *notion* of the magnitude of the aggregate designated by the *particular* number (and "particular" is emphasized). Thus, to say that a child knows the number "five" means that he has to be able to picture the appropriate "magnitude" of an aggregate. Here again the emphasis is placed on the idea that number is an *immediate* characteristic of an aggregate, a direct, *visual* property of it.

This methodological requirement can be seen most plainly in the following example. As a part of their study of the numbers up to 100, children may tie up 100 real matches in a bundle which, if done, apparently gives the child a "visual" notion of the magnitude of the number "100" (see [8:133]).

Number is understood here as a direct abstraction of a certain immediate property of an aggregate, the "volume," so to speak; the quantity of individual elements constituting it. Clearly, the means by which such an abstraction is achieved, according to the requirements of the classical sensationalist theory of abstraction, should be to compare *many* aggregates by the "volume" of elements in them. That is, if an individual distinguishes what is *common*, or *identical*, in the aggregates, then that is his "abstract" notion of the number of individual elements. This is exactly how the textbook says to assign a number to an aggregate, from the very start. Thus, a group of boys will be compared with a group of bicycle wheels, a group of sticks, and a group of dots. What can these aggregates, so different in nature, have in common -- what can be identical about them? Nothing, except the number of individual abstract elements which go to make them up ("two" in this case). It characterizes the immediate property of "magnitude" possessed by all these aggregates.

The children are similarly introduced to all the numbers up to ten. In every case, the number emerges as the abstract definition of the "magnitude" of an aggregate, which they find by comparing its individual elements, its units, with the units of other aggregates.[1]

What results from this curriculum and its teaching methods is that many first-graders who are "good" at counting (by the ordinary standards) still identify a number (a set of units) with an actual aggregate. They make no distinction between what they are counting and the method of recording the result and are unable to choose bases for counting and are unable to go freely from one to another -- they do not understand that number depends on the base which is chosen. As a result these children do not acquire a full-fledged concept of number, and this has a negative effect on all their subsequent study of arithmetic. It has been observed in particular that such children have difficulty mastering operations on concrete numbers and understanding the connection between whole numbers and fractions.

The traditional approach to introducing children to numbers has even more serious negative consequences. In particular, we believe that such negative consequences include the defects in the traditional introduction of numbers that A. N. Kolmogorov has noted. (He was referring directly to shortcomings in the introduction to the concept of real number, but as we see it, these shortcomings are deeply rooted in the child's introduction to positive whole numbers):

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[1] This theory of abstraction was worked out most consistently at one time by German methodologists (Grube and Lai). The so-called "numerical figures" we have in our textbooks are an echo of this theory and the methodological approach associated with it.

> To see that the generally accepted system [of introducing number] is pedagogically defective, one needs only to observe the difficulties pupils have when they learn that the meaning of geometric and physical formulas is independent of the units of measurement chosen and when they study the concept of "dimensionality" in these formulas (from Kolmogorov's preface to H. Lebesgue's book, The Measurement of Quantities, 2nd ed., Moscow, Uchpedgiz, 1960, p. 10).

A natural question comes to mind: Might it be possible -- experimentally, at this point -- to develop in first-graders a concept of number which would serve as a full-fledged basis for the mental operation of counting?[2] In 1962-63, A. P. Putilina, a first-grade teacher at School No. 11 in Tula, attempted to do this in a study under our supervision. The mathematics teaching in this class followed a special experimental curriculum. The children spent the entire first semester prior to the introduction of number becoming acquainted with basic quantities (such as length, volume, and weight) and with methods of comparing them and recording the results in letter formulas of equality and inequality. They were introduced to the basic properties of equality and inequality and the conditions in which it is possible to go from equality to inequality and back.[3] And only during the second semester were they introduced to number. In this section the introduction of number is described according to the experimental curriculum and the results of the experimental teaching are given.[4]

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[2] In previous years P. Ya. Gal'perin and L. S. Georgiev carried out experiments in kindergartens for the same purpose, [5].

[3] The theoretical justification for this approach to structuring the course, as well as some results of the teaching done by the experimental curriculum, are spelled out in articles by V. V. Davydov [3] and T. A. Frolova [4] (see also section two of this chapter [the preceding article in this volume (Ed.)]).

[4] In her teaching, E. S. Orlova has experimented with a comparable way of introducing number which differs only in that it includes no detailed advance introduction to the properties of equality and inequality of quantities.

Even before academic work was begun in the first grade (and then as the work progressed, but with no special instruction), the pupils' knowledge of counting was tested. It was found that many children were in fact familiar with the numerals from one to ten and were able to say them in order, and some were even able to say them backwards. Many were aware of the possibility of using numerals to count objects, and did so when directly given a small group of objects (from four to seven) to count. At the same time many had a poor grasp of the relations between numbers -- "Which is less, five or nine?" they would ask. This was particularly true when the numbers were greater than ten (sixteen or eighteen? nineteen or fifteen?). It should be mentioned that before beginning the sections of the experimental curriculum which had to do with counting, all of the pupils had learned "naturally" the names of the numbers from one to twelve or fifteen, and even beyond.

The section of the curriculum pertaining to number was divided into a series of topics. We should mention that since our approach to introducing number was based on teaching the children to find the relationship between an object (a quantity) as a whole and some part of it (the measure), in a certain sense number and counting was introduced on the basis of the measurement of quantities. Actually, however, this was not measurement in the precise sense, for the latter assumes a fixed unit of measurement (which we did not have at first) and as a rule refers only to continuous objects (whereas the children were taught to look for this relationship in discontinuous objects as well). The relationship between counting and measurement is not analyzed here, but throughout the description that follows (as in the actual teaching process), we have found it convenient to use the terms "object being measured" and "measure" to designate objects and operations in the experimental curriculum.

In describing the instruction process and its basic stages, we shall not go into an analysis of its foundations. Davydov has discussed in detail the structure of counting and its relationship to number in a study [1] to which we refer the reader. Our experimental curriculum is the practical outcome of that theoretical analysis.

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At the end of January in the second semester of 1962-63, the pupils in the experimental class began their study of the following topics:

1. Solving problems that required a determination of the relationship between a quantity (the object being measured) and a measure; learning the special operation needed for finding this relationship (including working with the measure and learning the rules for designating the results), and depicting the relationship as a standard set of *physical units*.

2. Using the names of the numerals to designate the results of counting.

The first topic was somewhat unusual. Its purpose was to convey to the pupil the necessity for using numerals as special mathematical "tools." In the first lesson the children were given the problem of selecting a piece of wood, from several in the corridor, that would be the same length as a certain model. The problem had one condition. They could not take the model with them! What were they to do?

Another problem they were given was to pour the same amount of water into one jar as was in another jar (the jars differed in shape and diameter, so it was impossible to judge by the water level whether the volume was equal). How could they do it?

After facing a number of such situations, the children began to realize that matching was possible not only in a *direct* way (by holding up a model against the object, in a comparison by some attribute), but also in an *indirect* way. Through leading questions the teacher helped the pupils to discover the basic requirement for doing this -- choosing a particular measure with which to do the matching indirectly.

The children learned to draw a little block each time they applied the measure. The result would be small groups of blocks, and the children found that with these they could now pick out "the same size" stick from those in the corridor, or they could pour in "the same amount" of water. They rapidly mastered the technique of measuring by using the results of a previous measurement (that is, by using an aggregate of blocks).

It took only two lessons to master this topic. The teacher did not attempt to teach the children all the rules for using a measure right away, of course. The "fine points" of the technique came at later lessons. But even at this point the children learned that the block was to be applied only when the measure would completely fit onto the model (or material) or could be completely filled with water (when this is what they are using). Otherwise the remainder was not to be counted.

The teacher constantly emphasized the necessity for using the blocks to designate the results of measuring (the results of the search for the relationship, that is). The children all understood the purpose well. Some used other objects instead of blocks, since the teacher had called their attention to the possibility of "substitution." *Any individual objects* could be used here to depict the results of measurement.

Using a series of problems, the teacher then showed that it was also possible to choose any measure (within the limits of practical convenience). However, once a particular measure had been chosen, the subsequent work (of taking the measurement and measuring off the matching object) could be done only with it. This topic was aimed primarily at teaching the children from the very start to make a clear distinction among the object being measured, the measure, and the *means* used to designate the relationship between them. A set of physical units was the "embodiment" of these means, and the teacher made a special point of mentioning this at the very beginning. He would ask: "How many of these measures (he would hold one up) is there room for (in the object being measured)?" The children would point to the pile of blocks or group of other objects and answer: "This many!"

During this period the teacher would ask the children to "measure" discontinuous objects (such as a group of blocks or squares), sometimes with a compound measure (consisting of two blocks, for instance). In this case it was still necessary to mark off the *individual* block when applying the compound measure to the object. The aggregate of individual blocks (units) expressed the relationship between the group

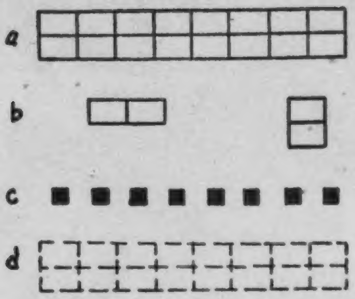

Figure 2. — Using a standard aggregate of physical units:
(a) the initial object, or model; (b) the bases for counting; (c) the results of counting — a standard aggregate; (d) the object reproduced according to the model.

of objects and the compound measure. With this aggregate and the measure, the children could obtain a new group equal to the first (Figure 2). The *distinction* was very clear here between *what* was being "measured," *what* was being used to "measure" it, and the individual *units* with which the results were shown.

The next topic included the replacement of the physical units by the names of the numerals and a more careful study of the relationship between the measure and the part of the object being measured, with the concept of "one" then being introduced.

The justification given for introducing numerals was that "blocks as units" are not very convenient (they can get jumbled; sometimes you need a great many of them; you cannot put them in writing if you need to give the results of measuring, and so on). The teacher showed the pupils that the rules for working with measures all remain as before except that instead of using blocks, one says "times." But this way we do not know how many "times" the measure has been applied -- one can say "times" endlessly, after all (one may see here the distinction between the temporal development of units -- since the word here is "unit" -- and the spatial development). These "times" have to be distinguished; *one* time, *two* times, *three* times, and so on. Or one does not have to say "times," but simply one, two, three, and so on, keeping in mind that each word denotes an application of the measure to the object. The last word ("seven," for instance) tells how many measures the object will hold. To solve a matching problem one must take this measure again seven times, measuring and saying the words in the proper order until one comes to "seven."

By performing a series of exercises, the pupils rapidly mastered the rules for using numerals (they had previously learned the names of them and the order in which they came). In some instances the teacher would ask them to substitute blocks or sticks for the words, so the children would shift back to a physical set of units. This time, though, they were counting the blocks and sticks (they would say "five," for example). Then the teacher would ask new questions. "How did you get these blocks? Why did there turn out to be five of them? What do these blocks tell you?" The precise answers would follow.

Special attention was given to carefully observing the rules for using compound measures and for working with discontinuous objects. At one lesson, for instance, sixteen blocks (laid in a row) were selected as the object whose length was to be measured. Three blocks in a row served as the measure. Using this measure, the children obtained the number five and a remainder. They themselves came to the conclusion that a separate block should not be counted -- "We can't say it's six -- that wouldn't be going by the measure."

Here is an excerpt from the record of the lesson January 30, 1963:

Teacher: Take two sticks (each 10 cm. long) and lay them together this way on your desk -- end to end (he shows how to do it on the board, as in Figure 3). This will be your object to be measured. Here is the measure (5 cm. long); hold it up (he checks to make sure everyong has the right measure). Count to yourself the number of these measures the object will hold.

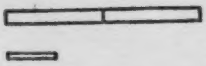

Figure 3.

Sasha S: Our measure fit four times!

Teacher: Right. Now show what two out of the four will be, going by our measure.

Sasha B., Lena P., and Madya M. hold up both sticks, the whole object to be measured; the rest of the pupils do the task correctly.

Teacher: Lena, show the children two, going by our measure.

Lena P. hastily takes away one of the sticks.

Teacher: Show them how you did it to begin with. Is that the right way, children?

Pupils: No!

Teacher: Why is it wrong?

Natasha P.: Because that's four!

Teacher: Going by what measure?

Natasha P.: This one (she holds up the 5cm.-long stick).

Teacher: Show us one, going by our measure!

The pupils grasp the lower part of the stick in their fists, covering it with the fingers of their other hands, and hold up half the stick.

Teacher: Now you have the same object to be measured, but this is the new measure to use (10 cm. long). How many times will it fit into this same object you are measuring?

Sasha B: I get that it'll fit two times.

Teacher: What about you, Olya?

Olya N: Two times!

Teacher: Now do you make sense out of that? First it went four times, but now it goes twice.

Lena P.: The measures are different!

Teacher: What kind of measure did we have the first time?

Pupils: A little one!

Teacher: And how many times did it fit?

Pupils: Four!

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Teacher: And the second time?

Pupils: We had a big one!

Teacher: And how many times did it go?

Pupils: Twice!

One can see from the report that in the learning process some children had "run together" two points of reference. On the one hand, they had been shown the necessity for especially finding the base for counting and for focusing on this base as they counted, rather than on the individual objects. On the other hand, these children still gave evidence of focusing on the individual object as the thing to be counted. This was apparently a part of the child's previous experience. As a rule, however, this orientation toward the individual object did not persist. Usually the child would correct himself. "Oh, that's wrong! This is what our measure is!" Then he would hold the measure up and count by using it as the base. Some relapses continued to occur during the subsequent work on counting, however. This indicates that these children had some difficulty focusing on whatever base was given and learning to stop using their former method of counting "individual items."

Particular attention was then given to "bringing out" the meaning of the concept of "one." Through special exercises the children were shown that an object to be measured may first be broken up into parts, each of which is to be equal to the measure, and then these parts counted. Each part will be "one," although it may itself consist of smaller elements. The children were shown that the content of "one" changes as the measure is changed, and thus that the total number of parts will be different. They practiced finding "one" for any measure they were given.

Here is an excerpt from the record of the lesson February 3, 1963. As usual, each child had sets of sticks, blocks, and mugs of various volumes in front of him.

Teacher: Put ten blocks next to each other in a row. This chain of blocks is our object to be measured. Hold this up as your measure (he picks up two blocks). How many of these measures are there in this chain?

Sasha B: Five!

Teacher: How many do you get, Natasha?

Natasha K: I got five, too.

Teacher: What is five?

Larisa T: That means the chain of blocks holds five of these measures of ours!

Teacher: Show what "one" is, going by our measure!

Some pupils "got stuck"; one girl immediately held up two blocks; Vitya held up one; in four seconds the majority of the children were holding up two.

Teacher: Serezha, show us "one" according to our measure. Work carefully and don't hurry.

Serezha P. holds up two blocks.

Teacher: Would it be right to hold up a single block?

Pupils: No!

Teacher: Why?

Andrei: Because it isn't equal to our measure.

Teacher: Galya, why do we have to hold up this many blocks?

Galya: Because we have to show what "one" is, one piece, what our measure is.

Teacher: How many of these measures did the object we were measuring contain?

Larisa: Five!

Teacher: Vitya, what were you supposed to hold up?

Vitya: One block.

Teacher: Is he right, Olya?

Olya: No, this is the measure we were supposed to show (she holds up two blocks; Vitya also picks up two blocks).

Teacher: Now this time this is our chain (five blocks). It is the object to be measured. And here is our measure (one block). How many times will this measure go into the chain?

Nadya: It'll go five times!

Teacher: Why does it come out that way? First you had one kind of measure, and it went into the chain five times, but now you have this other measure and it also goes five times?

Slava: The first time the chain was big and so was the measure, but now the chain is little and so is the measure!

Teacher: Now take one from our chain, going by this measure (one block).

Vova holds up two blocks.

Teacher: Vova, show everybody what you have. Is he right, children?

Vova wants to take away one of the blocks, but the teacher won't let him.

Pupils: No!

Teacher: Why not?

Borya: They don't equal our measure (Vova takes one block away).

Teacher: But why did you hold this up (two blocks) the first time I asked you to show me "one," and the second time I asked you the same thing and you held up this many (one block)?

Natasha: First we had a chain and the measure was this many blocks -- one, one (she holds up two blocks).

Teacher: And then what?

Yura: We took this kind of measure (one block). Here is one!

Teacher: You see, children, if I don't know what the measure is, then I can't say what "one" is equal to.

Four blocks arranged in a square were also used as a measure at this same lesson. The children laid out "one" by this measure, and then "one" more by the same measure. When asked, "How many are there altogether?" they answered, "Two," even though visually they perceived eight individual blocks.

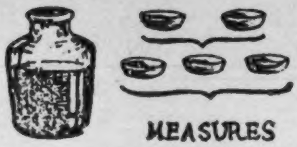

Figure 4.

Water was used frequently as the object to be measured, and compound measures -- two and three cups -- were chosen (Figure 4). The children were able to find "one" in either case with no errors.

Then they were given the following task as a test. The object was the word "Masha" written on the board. First the word was designated as the base for counting, then the syllable, and finally the letter. All the children accurately answered "one" by the first base, "two" by the second, and "five" by the third.[5]

At the next lesson the children applied various measures, "simple" ones and "compound" ones, to a single object. They learned to "combine" and "separate" the elements of the object when working with measures which did not coincide with these elements. For instance, at the teacher's request they laid out two squares of four blocks each on their desks.

Teacher: This is the object to be measured. The measure is a row of blocks (he draws a "pair" of blocks on the board; Figure 5a).

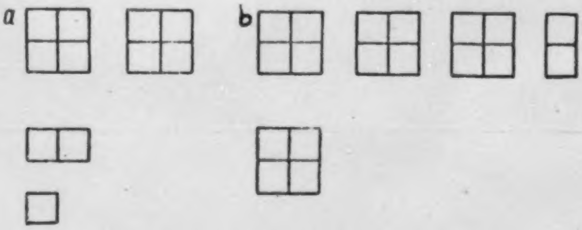

Figure 5. -- Changes in the number assigned an object as the base for counting is changed.

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[5] The children themselves used the word "measure" to designate these bases. We recognize that the term is somewhat inadequate here, but we are unable to provide a more accurate one at present.

How many times will our measure fit here?

Pupils: Four!

Teacher: Now with the same object to be measured but a different measure (a single block).

Pupils: Eight!

Teacher: Why did you get different numbers?

Vitya: Because the measures were different!

Teacher: Now add more blocks to the objects to be measured, the way the drawing shows (Figure 5b). Here is the measure. How many of these measures are there in all your blocks?

Pupils: One..., two..., three, and then there's a remainder.

Sasha shows how the remainder was obtained.

Teacher: Why are you calling these blocks a remainder, though, instead of counting them?

Vitya: Because less than a measure was left; it doesn't equal the measure!

Teacher: You already know how to count different objects by different measures now. What do you have to know, what do you have to keep in mind, so as not to make a mistake when you are counting?

Serezha: The measure!

At special lessons exercises were given in which the measure was either shown in a drawing (which was then promptly erased) or explained orally. The children would have to envision the measure as they counted. And since the measure changed from problem to problem, each child needed to exercise great care in determining the part of the object which was to be designated as "one." Almost all the pupils were able to handle these problems without mistakes. Particular attention was devoted to counting some "natural" object by different measures. For instance, using the classroom, the children were to count: "How many pupils are there altogether?" "How many boys? girls?" How many seats are there (not all the places were occupied)?" "How many desks?" "How many rows?" and so on. The children found here that each instance yielded a different number.

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In nine lessons all the pupils learned to count by whatever base was indicated or might arise in a practical situation (the counting far exceeded the limits of ten, as a rule). The pupils were introduced to numerical form and could move freely from one base for counting to another, understanding as they did so that the results of counting depended on the relationship between the object being counted and the base.

The task of the next topic (lasting ten lessons) was to show the child the possibility of working within the numerical series itself, the general principle behind it, and some principles of movement along it. To do this it was necessary to divorce the sequence of numerals from the direct counting of specific objects, that is, to give this sequence a logic of its own. We felt that this could best be done by charting the numerical sequence on a straight line. This work included the following stages; (1) teaching the children how to mark out numbers on a straight line; (2) teaching them how to form the "succeeding" or "preceding" number for any one given (by the principle n ± 1); and (3) teaching them the method of adding and subtracting numbers.

In the first stage, as the children counted different sorts of objects, they would obtain numbers. The teacher called their attention to the fact that no matter what objects were being counted and no matter what measures were used, the numbers one obtained were "identical" (5 here and 5 there; 15 here and 15 there). And in every case, to "get" to 5, for instance, one had to go from 1 to 2, from 2 to 3, from 3 to 4, and finally, from 4 to 5.

The teacher explained to the children that now they were going to "see" where these numbers could "live," how they were arranged, and how it was possible to "get" from one number to another. Then he showed that the numbers themselves could be arranged on a straight line or ray (the children fully understood this term). But to do this it was necessary to know certain rules. So the entire class, together with the teacher, "deduced" these rules, using the knowledge they had gained previously about the method of forming the numbers themselves.

Even earlier the children had learned the concept of "one" as the designation for the part of an object equaling the measure and thus unstable in its content. They understood, therefore, that "one," that is, the first step along the line, could be selected arbitrarily. When the teacher then asked, "Where is a place on the line for the number two?", and proposed several points which clearly did not correspond to "two," many of the children (including Serezha K., Misha P., Sasha S., Yura S., and Kolya Ch.) guessed that not just any line segment could be marked off for the number "two." Serezha K. expressed the idea this way. "You have to take a piece that's the same as one!" All the children then found the correct place for "three," "four," and so on. Thus when asked, "How far do you have to move from the number-four point to find the point for the number five?" Borya K. was able to answer, "As far as from zero to one or from one to two . . ."

After the children were given the rules for designating numbers by points on a line, they were shown that there is no limit to the possibility of moving "to the right" (any number can be marked off: twenty-six, or a hundred, or a million).

Then they were given exercises demonstrating that when an object is measured with one measure, one number is obtained (the number "three," for example). But this number can be put in various places on a line, depending on what "step" we choose for the number "one." The children would choose various "steps" and then find the places for the same number on different lines (Figure 6). They had no particular difficulty

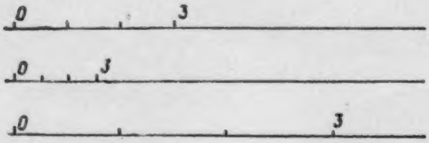

Figure 6. -- Charting the same number on lines when different "steps" have been chosen.

explaining "why" the position of "three" varied from line to line (they referred to the different "steps").

At the final lessons, as they charted numbers along lines, the children established that the smaller the number, the closer it is to zero, but the bigger it is, the farther it is from the zero.

The following is an excerpt from the lesson on February 8, 1963.

Teacher: I have a jar of water here. These two little jars together are the measure. I'll measure, and you count to yourselves. Mark out the number on a line.

Pupils: There were five (they mark out this number on a line).

Teacher: What number do we start from to mark out the numbers?

Lenya P: From zero!

Teacher: Which end is zero on?

Tanya Z: The left end!

Teacher: What might the first step be like, from zero to one?

Vova M: It can be any size we want . . .

Teacher: And can we also make the other steps any size we want to?

Borya K: No, we have to mark off as much for the other numbers as we did for the first one.

Teacher: And if we got seven instead of five, where would we put it - closer to zero than five is, or farther from it?

Lenya P: Farther from zero than five is.

Teacher: Why?

Lenya P: Because seven is bigger than five . . .

The second stage of the work on this topic consisted of introducing the formula for determining the number following a given one and the number preceding it. First the children were given the terms preceding and succeeding: "preceding" meant "coming directly before," and "succeeding" meant "coming directly after." At the teacher's request, the children would name two numbers that came next to each other, mark them off on a line, and determine which of them was preceding and which succeeding. Then they would find that the preceding number was closer to zero and the succeeding one farther away from zero.

After this the teacher asked them to determine, for several different numbers, how much more the succeeding number was in each case. The children came to the conclusion, after comparing the succeeding number with the original one in each case, that it is a unit larger than the original one, and that in order to obtain the succeeding one it is necessary to *add* a unit to the original number.

Up to this point the children had been working with separate concrete numbers. They needed to be introduced to the representation of numbers *by letters* as well (they had been familiar with letter symbols since the first semester, when they were recording the relations between quantities). This entailed special work, which may be outlined as follows.

The teacher asked the pupils to mark the number three on a number ray, with a "step" being equal to four squares on their notebook paper (they used red pencil to mark the point corresponding to this number on the line). Then they were given a new problem. "Find what number will be at this same point if we change the step and make it equal to two squares." With the help of the teacher, who was working the same problem on the board, the pupils all found that the number six would come at the "red dot." And they readily found the reason for the change in the number -- the "step" had changed.

In the next problem the "step" was different again -- six squares this time. And the red dot now corresponded to a different number. Together with the teacher, the children found it -- it was the number two. Again the "step" was changed, this time to equal one square. The majority of the children were able to find the number twelve on their own by this time and put it by the red dot.

The teacher called the children's attention to the fact that the *one* "red" dot depicted *various* numbers. *One* number could be replaced by *another* if the "step" were changed. The pupils performed exercises in finding the "step" so that the particular dot depicted a different number (such as the number four, in this case).

They worked with various rays, on which different points were marked with colored pencils, and by changing the "step" they were to assign different numbers to these points (one point thus corresponding, say, to the numbers ten, two, five, and one).

On the basis of this work the teacher was able to ask the pupils, "How else can you get a new number?" With complete self-assurance they indicated the method -- select a new "step." If they had already taken one square as a "step," then they could take half a square and get a new number that way. The teacher helped them to see that an even smaller "step" could be picked and that the number would then be even bigger. Any "steps" could be chosen. And the children had no particular trouble realizing that any number could be the result. In particular, they had a lively discussion with the teacher about what the "step" might be if the dot signified "a million." They knew that this was a very large number, and they were able to get a notion of the "step" by themselves -- it had to be a "wee little tiny" one.

As a result of this preparatory work, each pupil had in his notebook several number rays on which various numbers were assigned to different points. The teacher used these to show the children that all these numbers could be replaced by one letter -- it would tell about any number. If a point on a ray were selected and designated

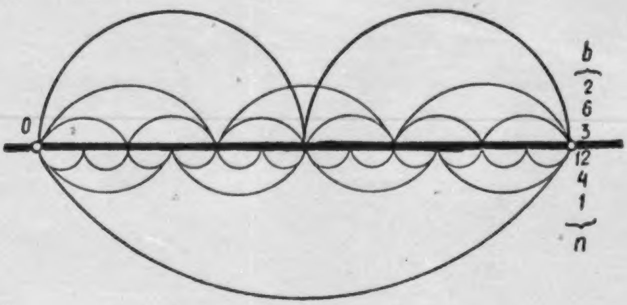

Figure 7. -- Diagram of the shift to using letters to describe numbers: b (or n) = 3, 6, 2, 12, 4, 1, and so on.

by a letter, it could stand for any number we want, depending on the "step" we chose (Figure 7).

This approach to letter designations did not cause the children any particular difficulty. They started doing exercises on replacing the letter symbol with specific numbers (by changing the "step," they would find that point A was equal to 1, 2, 4, and so on). In special problems they would replace the "many" numbers put down next to a point with a single letter symbol. They would use the most diverse letters here (such as A, B, n, m, and 1).

Then the teacher began to use letter symbols in exercises on determining "preceding" and "succeeding" numbers. There were such problems as: "What has to be done to the number n in order to obtain the succeeding number?" The majority of the children guessed right away: "You have to add one to n." For a few, however, the method of forming the succeeding number by moving "once more" was not entirely clear. They needed individual supplementary explanations and exercises in forming the succeeding number by the formula n + 1 before they were able to grasp the meaning of it. After this the children rapidly deduced the method of forming the preceding number (n - 1), with guidance from the teacher (Figure 8).

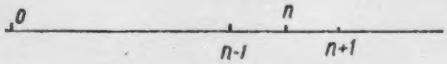

Figure 8.

Then they determined that the difference between adjacent numbers is always equal to one. In order to obtain each new number adjacent to the one they already had, the children would move one "step" to the right or to the left on the line, that is, add or subtract one. For instance, they were told to find the point on a line for the number four, then to find the point for the number which was one greater than four, and then to write the formula for finding it and determine this number. The children said that the new number could be obtained according to the formula 4 + 1 and that it was five. Then they found the number six and wrote this: (4 + 1) + 1 = 6. Then similarly they located the points for the numbers seven, eight, nine, ten, and so on.

In order to judge how well the children were learning the material in this topic, we gave them a quiz consisting of the following problems, at the beginning of the lesson on February 11, 1963 (it took twenty-five minutes).

1. The children were given an object to be measured and a measure (a piece of wood). They were to measure the length, express the results as a number, and mark off this number on two lines, with a different first "step" in each case. All the children solved this problem with no mistakes (100 percent solution).

2. The children were asked to mark off the number five on a line, and then the number preceding it and the number succeeding it. Then they were to determine what each of these numbers were. There was 100 percent solution on this one as well.

3. The children were asked to mark off a number K on a line and then the number that was a unit less and the number that was a unit more. Eighty-four percent of the children solved this problem. The mistakes were of the following types. Natasha G., for instance, marked off K and K + 1, but even though she wrote K - 1 in the proper place, she did not make the corresponding mark on the line. After the quiz, however, when she was asked, "Where should the number K - 1 be marked?" she pointed out the correct place. Olya B. "carried over," so to speak, the conditions of the preceding problem to this one. She placed the number K the same distance from the end of the line segment as the number five had been in the preceding problem, and she marked off the points for the numbers which were one unit less (four) and one unit more (six). Most of the other children who did not succeed in solving this problem made the following mistake. Having arbitrarily chosen a point for the number K, they would then try to find the points for the new numbers by starting from the end of the line segment as they would to determine the position of a known concrete number. Several weak pupils thus had difficulty learning to "find" a point for a number chosen arbitrarily.

4. The children were asked to mark off number P on a line, then the numbers that were four greater and three smaller. Eighty-one percent of the children solved this problem. Mistakes were of the following types. Two pupils accurately marked off P and P + 4 but omitted P - 3 completely. Two others wrote P + 4 and P - 3 but marked off identical line segments to the right and to the left of the number P. One pupil made the correct number of steps along the line (four to the right of number P and three to the left) but made a mistake in what he wrote down (he put "- 1" instead of "- 3").

The overall performance on these problems indicates that the majority of the children had learned that the results of measuring any quantities can be expressed as corresponding points on a line. They had learned the principle of forming numbers by moving along a straight line.

The next topic included the study of the interrelations of a quantity (that is, the object), a measure, and a number (the children would observe the relationship between measures when the quantity of the object was held constant and the numbers were varied, or they would observe the relationship between the magnitudes of objects when the measures remained the same but the numbers were changed, and so on). They were given problems in which they had to count using different bases (different measures, that is). When they gave the answers they had found (different numbers), the teacher asked, "Why do you get different numbers when you count, some big and some small?" Here is an excerpt from the lesson on February 21, 1963, when this topic was being worked on.

Pupils: (answering the question). Because we were using different measures.

Teacher: When you got the number six, what kind of measure were you using?

Natasha P: A little one.

Teacher: And when you got three?

Tanya Z: The measure was bigger.

Teacher: With the bigger number, what was the measure like?

Nadya M: Smaller!

Teacher: And with a small number, what kind of measure was it?

Zhora T: A bigger one!

Teacher: There is a special way to write all of this, children -- this way (he says it and writes it): $A_e = 6$. The object can be designated by any letter, A, for instance, and so can the measure, and we write it here (he points to it).[6] What we have written means that when e is the measure, object A equals six. Repeat this!

Pupils: When e is the measure, object A equals six.

Teacher: What have we just been doing?

Andrei D: Measuring a stick.

Teacher: What were we measuring it with?

Borya K: Measure e.

Teacher: What number did we get?

Lena P: Six.

Teacher: How shall we write this? Read it, Vitya!

Vitya M: When e is the measure, object A equals six.

Teacher: Then what object were we measuring?

Sasha B: The same object A.

Teacher: Let's designate the measure by the letter g. Now how should this be written?

Borya K: When object A is measured by measure g, it equals three.

The teacher writes $A_g = 3$.

The pupils write in their notebooks:

$$A_e = 6$$

$$A_g = 3$$

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[6] The next year this relationship was immediately given in the form: $\frac{A}{E} = 6$.

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addition and subtraction (expressed in direct and indirect forms), the two ways of comparing numbers (by subtraction and by division), the two ways of decreasing or increasing a number (by several units and several times), and others.

Taking this into consideration, the second grade curriculum stipulates that the children solve a large number of appropriate simple arithmetical problems which reveal the essence of the distinctions on which the differentiation of these concepts is based, using material which is concrete and close to the children's ideas and interests. Work on problems which are very diverse, not only in content but also in arithmetical essence, affords wide possibilities for further perfecting and deepening the knowledge, skills, and habits the children acquired in the first grade.

Along with the development of separate, individual skills necessary for independent problem solving (the ability to read the problem, illustrate its conditions, to pick out the data necessary in order to answer the question, to outline a plan for solution, etc.) in the second grade, the next step forward, in the simultaneous use of these individual skills in solving not only simple, but also compound problems, must be made. The curriculum stipulates the instruction of children in the second grade in the solution of problems in two or three operations, including all the types of simple problems with which they dealt in the first, and then in the second grades.

In determining the goals of instruction in the second grade, it is necessary not only to consider the curriculum for this grade, but also to think about the goal for which the teacher must prepare the children in the first two years of arithmetic instruction. With this approach it becomes clear that the most important task in the second grade (aside from those enumerated above) is to create conditions under which the children amass knowledge of a number of arithmetical facts, necessary for the generalizations stipulated by the third- and fourth-grade curricula. This requirement must be reflected both in work on problems and in work on examples.

Indeed, aside from the significance of the solution of examples in the development of computational skills, which was shown above (see the section devoted to the various types of exercises dealing with the

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solution of examples), work on examples affords broad possibilities for preparing the children to understand the relationships between separate arithmetical operations and among component operations, for acquainting the children with the composition of numbers from addends and factors, and with the laws of arithmetical operations. Account must be taken of all these conditions in the development of a system of children's independent work in arithmetic lessons in the second grade.

It follows from the goals formulated above, above all, that the basic content of children's independent work in the second grade must be the solution of arithmetical examples and problems (not only simple, but also compound) in order to develop the appropriate skills and habits. A place, moreover, must be set aside for exercises directed toward a deeper study of the features of the arithmetical material with which the children must deal. Below, we consider the concrete forms in which these requirements are realized in the study of the primary topics of the second-grade curriculum.

Pupils' independent work in lessons on the topic "The Four Operations within the Bounds of 20." The present topic is devoted to the review of what was studied in the first grade. Much attention must be given to reviewing the tables of addition and multiplication within the bounds of 20. It is also very important to freshen the children's memory of the devices and methods of computation with which they were acquainted earlier and the devices and methods dealing with problem solving. As always, the review must be organized so that it facilitates, to some degree, the enrichment of knowledge acquired earlier, and the perfection of the skills and habits just formed.

Pupils' independent work must occupy a relatively large place in the review lessons. Along with exercises of types well known to the children from the first year of instruction, it is useful to introduce several varieties so that, in executing the teacher's assignments, the children must look at the same material from another point of view. For this reason, aside from the usual training exercises dealing with the solution of prepared examples and problems, it is especially important to make use of assignments requiring a great deal of independent thought and initiative from the children. Thus, in reviewing addition and subtraction with and without carrying over ten, the assignments requiring the children to compose examples from a given model prove to be very useful. Models for these assignments are made so that the children, while executing the assignments, receive material for the composition of various instances of the operation. In our experiment, for example, the children were asked to compose two more pairs of examples from the model:

6 + 3 =	7 - 2 =
16 + 3 =	17 - 2 =

After solving the given examples and independently composing analogous ones, the children were asked to be prepared to explain the solutions they had reached. As a check, the teacher asked how the examples in each pair were alike and how they were different.

The children's independent construction of examples from a given answer is also frequently used as a review. For example, they were asked to compose any six examples with 18 for an answer. In this case, it depended on the pupils' own initiative whether they made up only examples which did not require carrying over ten, or whether they used numbers for which the ability to add and subtract carrying over ten was required. It also depended on the students' own initiative whether they used, let us say, only addition, or included subtraction as well, and finally, whether they composed examples on multiplication. This assignment can be given during the review of addition and subtraction without carrying over ten. However, by the way the children approach it, the teacher can tell approximately how well each of them remembers other instances of the operations from the first grade.

Further, because in future work in the study of addition and subtraction within the bounds of 100 the children's reasoning must often proceed analogously their reasoning in the study of addition and subtraction within the bounds of 20, we included, as early as the first weeks of the classes devoted to reviewing what had been covered, assignments which served as a certain preparation for such reasoning. The children were asked to compose examples analogous to the ones in the model, which used the first ten numbers, but using numbers beyond 10.

For example, as a model the teacher gave the children examples like: 3 + 5 = 8, 5 + 3 = 8, 8 - 5 = 3, 8 - 3 = 5. From this model they had to compose analogous examples with the numbers 18, 12, 6 and 20, 6, 14. Then the children were asked to independently compose any example in addition and then construct the corresponding subtraction example.

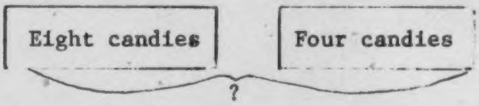
Such exercises represent a development of the work conducted in the first grade. They lay a wider foundation for the formation of the proper generalizations (about the link between addition and subtraction, and about the interdependence of the components of these operations); and they are good material for practice in drawing analogies. In drawing an analogy, in this case, the children must apply a regularity which was observed in smaller numbers to work with larger numbers. This kind of analogy is precisely what is necessary in preparing for the kind of reasoning which later must be relied upon when considering operations within the bounds of 100.

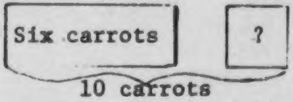
In the first lessons in the second grade, it is already quite possible--and very useful--to give the children practice in independently making comparisons by juxtaposing a pair of examples which differ by only one feature. In selecting examples for exercises, it is necessary, of course, to strive for the condition so that the conclusion which the children can reach through comparison acquires some cognitive meaning, i.e. deepens the knowledge which they have acquired earlier and serves as preparation for the following work. The following is a model that can be used to create the foregoing condition. Two examples are written on the board: 18 - 2 and 18 + 2. The teacher asks the children to solve them, to think about how they differ, and to explain why, in the solution of one example, the answer is greater than 18 and in the solution of the other it is less. In the check, the pupils explained that in these examples, the numbers are the same--18 and 2, but in the first it is necessary to take away 2 from 18 and in the second to add 2 to 18; that if 2 is taken away from 18, the number is smaller, and that if 2 is added, the number is larger than it was.

Not only do such exercises develop the children's powers of observation and capacity for the analysis and understanding of causal relations; they also help to deepen the knowledge of arithmetical operations which the children acquired in the first grade, where this deduction was not made in a generalized form.

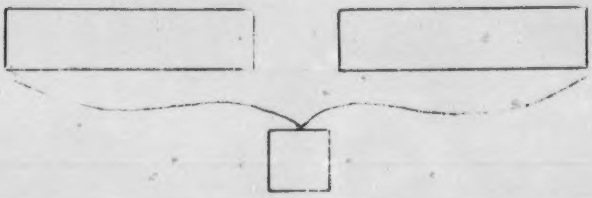
Independent work on problems in the review of what was covered in the first grade must also be directed not only toward freshening the children's memory of what they learned in first grade, but also toward deepening this knowledge. Thus, in the first grade sufficient attention was given to the diagrammatic notation of the conditions of a problem on finding the sum of two numbers and on finding one of the addends from the sum and the other addend. The children, for example, knew how to make diagrams for problems of the following type:

> In one box there were eight pieces of candy, and in a second, four pieces. How many pieces of candy were there in all in the two boxes? There were 10 carrots in two bunches. In one bunch there were six. How many carrots are in the second bunch?





After reviewing with the children the notation for problems of this type and also the composition of a problem from a diagrammatic outline, the teacher may give the children a pair of these problems for independent work with the assignment to write both problems according to the following diagram:



The children must not only independently apply the familiar method of the diagrammatic representation of conditions, but must also unwittingly perceive the difference between the problems under consideration—a variation which requires the differential use of the same diagram and leads to different solutions.

In reviewing problems on increasing and decreasing a given number by several units, it is also very useful to construct assignments for the children's independent work so that, from the very beginning, the children compose and compare, in the course of doing them, corresponding pairs of concepts.

Exercises analogous in nature can also be carried out in a review of multiplication and division within the bounds of 20. Most of the pupils' attention must be directed toward reviewing the meaning of these operations. For this reason, both in solving problems and in reviewing the tables, it is useful to organize the children's independent work so as to deal with the illustration of a problem's conditions, and to reveal the meaning of an operation (replacement of multiplication by addition, and vice versa). It is best to organize the check of the mastery of the tables in the form of an "arithmetical dictation."

After the review of what was studied in the first grade, the children turn to the study of numeration and the four operations within the bounds of 100. We will consider the primary units of this topic.

Numeration and the four operations with whole numbers of tens. The children were acquainted with numeration within the bounds of 100 at the end of the first year of instruction, so this question must, on the whole, be considered as a review. What is new to the children in this topic are the operations with whole numbers of tens, and problems in two operations including multiplication and division.

The use of visual aids is very important in understanding operations with whole numbers of tens. Using counting sticks tied in "bundles" of 10 each, the teacher must make the children conscious of the fact that 10 sticks constitute 1 ten, and 1 ten is nothing other than 10 sticks (units). After the children gain an understanding of this principle through visual demonstration and through work under the teacher's direct guidance, all the operations with whole numbers of tens can be examined on the basis of the children's independent work. The children's independent work is the starting point in the lessons devoted to the study of each new instance. The independent work is built on the material of the first ten numbers in preparing for the study of the corresponding instances of operations with whole numbers of tens. It is also useful to make use, in assignments, of material which affords possibilities for the composition and comparison of corresponding instances of operations. For example: 3 + 5, 30 + 50, 8 - 6, 80 - 60.

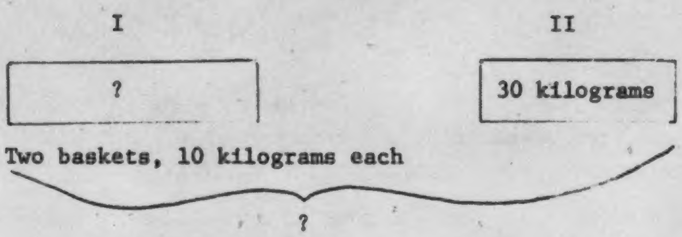
After the pupils consider, under the teacher's guidance, the illustrations and detailed notations given in the textbook to clarify new instances of operations with even tens, it is possible to ask them to try independently to gain an understanding of a notation relating to a new instance (for example, after they have already understood the addition illustrated on page 11 and the multiplication on page 15, division from a book can be used in conducting the children's independent work). The assignment may be given in this form:

> Carefully examine the solution to example 40 on page 17 of the textbook, show with the sticks all that is written there and be prepared to explain the solution of this example.

In exercises directed toward consolidating the acquired knowledge, it is important to include numerical material, not isolating operations with even tens, but combining work on them with other operations within the bounds of 20.

The possibility of using children's independent work when introducing problems of a new type, and of using their independent work on compound problems including multiplication or division was mentioned above. As preparation for solving such problems, one should review with the children all the methods and devices for work which they used in first grade for solving corresponding simple problems. Just before solving the new kind of problem, the children are asked to solve, in independent work, two problems analogous to those of which the new one is composed. After checking this work, the teacher can present the new problem, analyze its conditions with the children, explain that it is not possible to get the answer to the question at once, and then ask the children to solve it independently.

In some cases the diagrammatic notation, to which the children grew accustomed in the first grade, proves very useful. For example, in order to clarify to the children the method of solving problem 127 from the textbook, a diagrammatic representation (apart from the drawing in the book) is useful:



The diagram is made in the following way. The teacher reads the problem's text:

> Some schoolchildren gathered two baskets of apples with 10 kilograms in each basket from one apple tree, and, from another tree, 30 kilograms of apples. How many kilograms did the children gather from the two trees?

Then one of the children repeats the question and it is explained that they must find the kilograms of apples which were gathered from the two apple trees. Thus the diagram must have two boxes (as is done in the first grade in the solution of compound problems including the increasing or decreasing of a number by several units). The children are asked further, whether the number of kilograms of apples which were gathered from the first apple tree, and whether the number of kilograms of apples which were gathered from the second tree are stated in the problem. The appropriate data are written in the diagram (a question mark is put in box I, and "30 kilograms" is written in box II).

> What is stated in the problem about the first apple tree?

Again, the appropriate figure is written, but this time below the first box (as was done in the first grade, in the construction of diagrams for problems which require the increase or decrease of a number by several units). Finally, with the help of a bracket and question mark, it is indicated what must be found out in the problem.

After the conditions are analyzed and noted in the diagram, the children independently solve the problem. Later, in the solution of problems of the given type, one may begin to include in the children's independent work the diagrammatic representation of their conditions and the composition of problems from such representations. This work provides further development of the knowledge and skills acquired earlier, since the children learn to apply them under new conditions; this has great significance for instruction in problem solving.

Pupils' independent work in lessons on the topic "Additions and Subtractions within the Bounds of 100." This major topic requires approximately six weeks of class time. It is divided into two subtopics--addition and subtraction with, and without, carrying over ten. The study of new instances of the arithmetical operations is here interwoven with the introduction of new types of problems (problems in which it is necessary to increase or decrease a number by several units, indirect problems on finding an unknown minuend or unknown addend from the sum and the other addend, problems on finding the third addend, on comparing numbers by subtraction).

There is no major difference in the organization of children's independent work in the study of addition and subtraction both without carrying, and with carrying over ten, since both are equally familiar to the children from the first grade where they were studied using numbers within the bounds of 20. Hence, we will consider questions relating to the study of new instances of addition and subtraction as a group, and separately analyze questions connected with instruction in solving new types of problems.

The system for the study of various cases of addition and subtraction is very clearly defined in the textbook, which provides for a gradual shift from easier cases to more complex ones. The selection of numerical material for children's independent work should follow this system. Pupil's independent work in the study of each new instance of addition or subtraction should appear during preparation for the perception of new material, during this material's introduction, and during consolidation. Preparation for considering each new instance will most frequently consist of solving appropriate examples, using what was learned before.

For example, in the lesson on the introduction of addition, without carrying over ten, within the bounds of 100 (e.g., 45 + 3), the children may be given, as preparatory work, examples on addition within the bounds of 10, and also corresponding examples on addition within the bounds of 20, such as 15 + 3, 17 + 2, etc. It is also very useful at this stage of instruction to continue using practical exercises with visual aids. Here the same materials with which the corresponding instances of operations were explained in the first grade (counting sticks and bundles of sticks) are used. This makes it possible to demonstrate visually the similarity of the new cases to those which the children encountered working with numbers within the bounds of 20.

By gradually increasing the proportion of the children's independent participation in the study of new cases by analogy with familiar ones, it is possible, finally, to bring the children to the independent examination of new material as described above. This is relevant to addition and subtraction carrying over ten. Here it is useful to use visual aids analogous to those used in the first grade. There the device "The Second Ten," a demonstration board consisting of two rows of boxes with ten in each was used; here we propose the device described by G. B. Polyak called "Calculation Table. The First Hundred" [17:146-47].

In examining problems of the type 30 + 26 or 87 - 30, it is necessary, as preparation, to solve not only examples on addition and subtraction within the bounds of 10, but also examples on addition and subtraction with even tens. Since all the material which must be used in preparing for the study of the new topic is well known to the children, the teacher must try to construct assignments so that the independent work is not monotonous, using for this purpose various types of assignments dealing not only with the solution, but also with the children's independent composition of examples, which we described above. This is also true of exercises for independent work directed toward the consolidation of new knowledge. Especially significant is the use of assignments which require the children to make comparisons, establish points of similarity and difference between observed examples, and reason by analogy. The appropriate work is a development of what was outlined for the first lessons devoted to reviewing material already covered. Thus, so that the children may establish more precisely the similarity between cases of a single type of addition, using numbers of different magnitudes, it is possible to give the following assignment for independent work. Columns of examples are written on the board (it is even better if the corresponding cards are prepared for the individual work of each pupil):

6 + 3	8 + 2	7 + 5
16 + 3	18 + 2	17 + 5
26 + 3	28 + 2	27 + 5
36 + 3	38 + 2	37 + 5
...	...	...

The children are asked to continue these columns, constructing examples of the same type.

In checking the students' work, we established how the examples in each column differ from each other, and how the differences in examples lead to differences in solution. Thus, a general rule for the solution of problems of the given type is formulated. In completing the assignment, the children must not only perform the appropriate calculations, but also make comparisons between the examples they have solved; note the general principle by which they are arranged; independently compose, on this basis, the next examples; consider all the examples in each column as a whole; and draw a general conclusion about the method of solving them.

It is also useful to give, for comparison, examples in which the differences concern the method of computation. Thus, one column of examples may represent addition without carrying over ten, and the second, with carrying. In comparing these columns, the children must notice this feature, and themselves compose examples relating to each aspect.

All the examples carried out with material on the first twenty numbers in order to provide a deeper familiarity with the composition of numbers and properties of arithmetical operations, must be repeated with material on large numbers which the children first encounter in the second grade. The corresponding assignments will also be built around the transfer of earlier-acquired knowledge to a broader range of numbers (with the help of analogy). Some examples of such assignments follow.

Earlier the children did exercises in which they were required to indicate the composition of a given number according to a model:

17 = 10 + 7   12 = 10 + _____
14 = 10 + 4   16 = _____

Now the analogous exercise must be performed with the first hundred numbers:

$36 = 30 + 6$ $\qquad$ $58 = \underline{\qquad\qquad}$
$27 = 20 + 7$ $\qquad$ $43 = \underline{\qquad\qquad}$

In the first grade the children solved the so-called examples with blanks of the type $6 + \underline{\quad} = 8$ and others. Here, they can be given analogous examples with larger numbers: $26 + \underline{\quad} = 29$, $28 + \underline{\quad} = 30$, etc. Until this time the children used the commutative property of sums only with numbers less than 20. Now they can be given the opportunity to check it for larger numbers. With this purpose, they can be asked to compose examples from the model: $23 + 7 = \underline{\quad}$, $7 + 23 = \underline{\quad}$.

Solution of examples in two operations, as well as in one operation, should be included in the children's independent work. It is also useful to assign examples with one of the components left out. For example: $14 - 2 + \underline{\quad} = 15$. Various examples of this type can be introduced through exercises in the completion of "magic squares," which are perceived by the children as a kind of game and excite great interest. They are very useful for developing the skill of mental computation.

This gradual increase in the complexity of assignments dealing with the solution and composition of examples facilitates not only the formation of the proper computational skills, but also the children's deeper mastery of the methods of operation, properties of numbers, and relationships among the components of operation.

In instruction in solving new types of problems, the nature and place of the children's independent work depends on the characteristics of each type of problem. Several of the problems introduced do not cause the children any particular difficulty, since their solution is based entirely on what the children already know and requires only the application of knowledge and skills acquired earlier under somewhat altered conditions. In these cases, independent work can be given to the children from the very beginning at the stage of introduction. This was shown above, for example, in problems in which the increase or decrease of a number by several units was encountered twice.

In other cases, the examination of a new type of problem can be conducted through recourse to the children's acquired experience with practical operations with objects. This is true of problems on comparing numbers by subtraction. Here independent work can also serve as a starting point in the introduction of new material, but it will differ in nature from the preceding case. There the goal of independent work was to freshen the children's awareness of a series of arithmetical facts which they learned in the first grade, i.e., the realization of the knowledge and skills of problem solving, knowledge and skills which must be used in solving a new type of problem.

In a lesson devoted to the comparison of numbers by subtraction, we are not dealing with earlier-acquired knowledge applied under new conditions. The children do not yet have the knowledge which would allow them to independently solve a problem of this type. Here we only suggest that, in their practical experience, the children more than once have had to solve the problem of the comparison of two objects, that the very statement of the question may be familiar to them and that thus, if we use their practical knowledge, it will be easier to bring them to an understanding of the arithmetical essence of the problem.

Independent work preparing the pupils to examine a new kind of problem must thus be of a practical nature. The children can be asked, for example, to compare practically the length of two strips of paper, two tapes, etc. By performing the approximate practical operations, the children soon can understand what precisely must be determined in this type of problem, and what arithmetical operation corresponds to the practical operations which they used in solving the problem.

Finally, the pupils encounter problems which the knowledge they acquired earlier does not help to solve; the knowledge may even hinder the mastery of new material. We have in mind the so-called "problems expressed in indirect form"—problems on finding the unknown minuend from the subtrahend and difference, or on finding an unknown addend from the sum and other addend. Problems of this type have more than once attracted the attention of methodologists and psychologists. Their interest is determined by precisely this feature--that the children's study of new material is, in this case, in direct contradiction to what they learned before this time. Thus, although during the whole first year the children always dealt with problems in which the expressions "made in all," "brought more," "bought more," etc., invariably implied the operation of addition, and in which the expressions "gave away," "ate up," "was left," etc., implied subtraction, it will now be necessary, when solving problems on finding an unknown minuend or addend containing these same expressions, to apply the operations in a way opposite to that which seems to suggest itself to the child under the influence of previous experience in solving direct problems.

Keeping in mind the difficulties such a reversal causes the children, the teacher must, in this case, very carefully compose and prepare an explanation accompanied by visual material. (The most expedient form of visual aids for explaining to the children the process of solving indirect problems is dramatization, which permits the illustration not only of the components of operation, but also of the operations themselves; such illustration is especially important for problems of this type.)

Pupils' independent work can be used here only at the stage of consolidation, after the children, under the teacher's guidance, have gained an understanding of the special features of the new problems. Practical experience and special studies indicate that even after the children have understood the characteristics of these problems they continue, for a very long time, to make errors, confusing indirect problems with the corresponding direct ones.

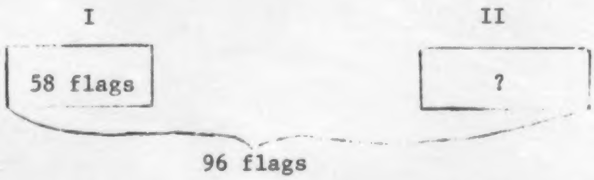
For this reason, when teaching children to solve indirect problems, it is very important to provide a selection of exercises for independent work which would afford sufficient material for discrimination and differentiation. For this purpose, it is useful, at this stage of instruction, to solve not only indirect problems, but the corresponding direct problems studied earlier as well. This excludes the possibility of solving problems mechanically, without sufficiently analyzing their conditions, or considering the specific characteristics of each type of problem.

However, one must do more than give direct and indirect problems alternately to the children for independent solution. It is also necessary to make sure that they have learned, when solving new indirect problems, to apply to the analysis of their conditions the devices and methods which they should have learned by this time, and also to see that they have mastered several new devices which prove especially useful in the solution of indirect problems.

In connection with this aspect of the solution of indirect problems, as work for the whole class and then as independent work, we successfully assigned diagrams and outlines of the conditions. An example is cited to illustrate how this work was conducted. The children were asked to independently solve the following problem, on finding one addend from the sum and other addend.

> To prepare for a holiday, the children made 58 flags in one day. The next day they made some more flags: there were 96 flags in all. How many flags did they make on the second day? (No. 336 from the second-grade textbook.)

The independent work was divided into two stages: (a) represent the problem's conditions by a diagram, and (b) solve the problem. The children were allowed to go on to the second stage of work only after the teacher had checked the diagrams of the conditions. The teacher conducted the check in the course of the work--walking up and down the aisles and looking over the pupils' notebooks. However, after making sure that all the children had been able to handle this part of the task, he submitted the task of checking the first stage of the work to general discussion. For this, one of the pupils was called on to write the problem's conditions on the board, explaining each step in his work. Other children on whom the teacher called participated in the explanation. The following diagram was written on the board:



The construction of the diagram was accompanied by an explanation.

> In one day the pupils made 58 flags--we will write that in box I. On the next day they made some more flags--since it is not stated how many, we must put a question mark in box II. Further, the problem says that there were 96 flags in all--that is how many they made in two days; we will draw a bracket and write down that in two days they made 96 flags. The problem asks how many flags did they make on the second day? We have a question mark in box II--we must answer this question.

After this analysis of conditions, the children solved the problem independently. It did not cause them any difficulty, since they recognized in the first diagram a problem of a type known to them since the first grade.

In the above case the diagram helped to indicate the general principle which unites problems on finding one addend from the sum and the other addend when they are expressed indirectly, and when the problem's formulation does not contain expressions which suggest the choice of one or another operation like ("In two bunches there are 20 radishes. In one there are 10. How many radishes are there in the other bunch?"). For this purpose the device of outlining the conditions was also used. In many cases, it facilitated the solution of indirect problems, since such a notation includes a whole series of separate expressions used in the complete text of the problem, and emphasizes the indirect nature of its formulation. An example is given as illustration.

> Asters were growing in a flower bed. The children picked six asters for a bouquet. After this, eight asters remained in the bed. How many asters were there in the bed at the beginning? (No. 266 from the textbook).

The outline of the conditions of this problem looks like this:

For the bouquet - 6 asters
Left in the bed - 8 asters
How many in all?

In writing the conditions of this problem one is half-way to solving it, since in this form it does not differ from problems well known to the children since first grade.

It is not very complicated to prepare the pupils for the independent execution of diagrams and outlines of the conditions of indirect problems if they have mastered these methods of representing various types of problems in preceding lessons. For consolidation and drill, it is continually necessary to include this type of assignment in independent work on new types of problems. Here again the knowledge, skills and habits which the children acquired in previous stages of instruction undergo development.

However, as was noted above, the solution of indirect problems is related to the use of still another way to approach the analysis of conditions, and the search for the method of solving a problem. We will deal with this in more detail.

The indirectness of formulation which hampers the understanding and solution of problems of this type is, in fact, still a formal indication; the problem's formulation may be changed so that the indirectness of formulation disappears, completly revealing the mathematical essence of the problem in the new formulation. An example illustrating this point follows.

> In a state farm, there were 16 tractors. When they sent some more tractors, there were 22 tractors in all on the farm. How many tractors did they send to the state farm? (No. 334 from the textbook).

In this formulation everything suggests addition to the pupil. Indeed--"There were, then they sent more...were in all.."--here not only are the separate expressions strongly associated in the children's minds, with the choice of this operation, but the course of the practical operation described in the conditions logically requires the performing of addition. As a result, even if the children correctly answer the question, they often write the problem's solution thus:

16 tr. + 6 tr. = 22 tr.

We will now formulate the same problem in another way.

> In a state farm, there are 22 tractors. Of these, 16 tractors were there earlier, and the rest were sent later. How many tractors were sent to the state farm?

We see that, from this rephrasing of the conditions, the essence of the problem does not suffer at all. Moreover the problem formulated in this way leaves no cause for doubt that it must be solved by subtraction (the children have encountered problems formulated in this way more than once even in the first grade).

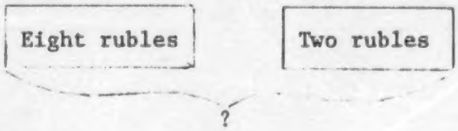
Accordingly, one of the devices facilitating the understanding and solution of indirect problems is this rephrasing of the conditions. An alteration in the formulation is one of the general devices which prove useful in the solution of other problems as well. In the work on the psychology of instruction which we have already quoted [3], this device is recommended as one of the distinctive means for facilitating problem solving. Thus, it is advisable, when the children are studying indirect problems, to acquaint them with this device, and to teach them to use it with awareness.

After carrying out the appropriate work with the teacher's guidance and help, the children may be assigned to change the formulation of a problem in their independent work on the conditions of this problem. A model of an appropriate assigment is cited: "Carefully read the problem and try to express it so that it is immediately clear how it is solved." The children must be given sufficient time to execute this assignment. Afterwards one should call on at least three or four pupils. The rest of the children should listen carefully to how they formulated the problem's text, and make suggestions for the correction and increased precision of the formulation. This task is the next step forward in instructing the children in the conscious reading of the conditions and their precise representation. The ability to express the same idea, the same relationships in a different form is one of the important indications of the pupils' development; hence such exercises have great educational significance.

In later exercises directed toward the consolidation of knowledge, skills, and habits acquired earlier, as we have said above, it is useful to include not only indirect problems, but also those directly-expressed with which the children confuse them. Here it is wise to formulate an assignment for the children's independent work which specially directs the pupils' thoughts toward the juxtaposition and comparison of these problems. So that this comparison may thoroughly reveal the peculiarities of these problems, one should vary the assignment, asking some of the children to diagram the conditions of both problems of the pair, allowing the children to establish the difference between them. In others, on the contrary, one should direct the work toward bringing out the similarities between indirect problems and the corresponding direct ones (as we showed above, with the example of the assignment of diagrams and outlines of the conditions). Finally, one should assign work in which the children must mark the points of similarity in comparing the formulation of indirect and direct problems, and underline the differences in the course of solution.

Along with these assignments it is constantly necessary to continue the work begun in the first grade, whose purpose is to develop in the children the ability to supply the question for data, to select the data necessary for answering a question, to compose a problem by analogy, etc. Here too in the exercises one may successfully follow the same principle on which the work on indirect problems was based. For example, it is possible to ask the children to compose two problems, one indirect, the other direct, from one diagram.

They are given the diagram:



The children are asked to compose one problem in whose conditions the words "were left" are used and another in whose conditions is the word "more." While checking the problems the children have composed, the teacher may ask them to solve both problems in the same way.

All during the work on the topic "Addition and Subtraction within the Bounds of 100," the children's independent work must consist of both the completely independent solution of simple problems of a type studied earlier, and the solution of compound problems which they solved in the first grade (using all the diverse forms of assignment used in the first grade).

Pupils' independent work in lessons on the topic "Tables of Multiplication and Division." This topic includes the study of all instances of multiplication and division by tables within the bounds of 100, and the introduction of various applications of these operations. Along with the construction of tables, their study, and practice exercises having as a goal the firm mastery of the tables of multiplication and division, much attention is devoted in this topic, as in the preceding one, to problem solving. Here the children first encounter problems on division according to content, on finding the parts of a number, and on increasing (and decreasing) a number by several times. They also encounter multiple comparison of numbers and problems solved by the method of reduction into units.

It is possible to regard the work carried out in the first grade as preparation for the study of multiplication and division within the bounds of 100. For this reason, here, as in the study of addition and subtraction, it often proves possible to prepare for and sometimes even to carry out the consideration of new material on the basis of the children's independent work.

Thus, as preparation for drawing up each new table, the children may be given diverse exercises on familiar material directed toward the review of the meaning of multiplication. For example: Problems requiring the replacement of addition with subtraction and vice versa, the continuation of an appropriate series of numbers (3, 6, 9, 12..., 4, 8, 12, 16 ...) to 100, and others.

During preparation for the construction of multiplication tables within the bounds of 100, the children can be asked to draw up independently the portion of the table which they learned in the first grade. For example, they can be asked to continue this table:

$3 + 3 = 6$ $\qquad$ $3 \times 2 = 6$

$3 + 3 + 3 = 9$ $\qquad$ $3 \times 3 = 9$

$3 + 3 + 3 + 3 = 12$ $\qquad$ $3 \times 4 = 12$

..... $\qquad$ .....

It is not worthwhile to set any limits in the assignment--experience shows that many pupils construct the whole table of multiplication by three themselves, and not just within the bounds of 20. If there turn out to be many such children in the class, the teacher may let one of them put the new portion of the table on the board, including the other pupils in this work as well. In any case, after the construction of the first two or three tables, the rest may be made on the basis of the children's independent work. The teacher need only check on whether all the children have really understood how these tables are constructed, and organize further exercises directed toward their mastery.

When new tables are introduced, the children become acquainted with several new devices for selecting various addends. To make sure that they master these devices based on the properties of multiplication, it is necessary to include appropriate assignments in the pupils' independent work.

For example, the teacher may ask the children to write one of the examples in the multiplication tables directly. Let us say the example 4 x 8 was given. This example can be written in another way, as follows: 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4, 4 x 4 x 2, 4 x 2 x 2 x 2, etc. A detailed notation of the calculation can be used for this same purpose:

4 x 8 =
4 x 4 = 16
4 x 4 = 16
16 + 16 = 32

It is possible to give this representation as a model and ask the children to write other examples from this model (7 x 6, 2 x 8 etc.).

To consolidate knowledge of a table, it is possible to use all the exercises of the same type that were used in the study of addition and subtraction--the construction of examples from a given operation and one of the components (construct four examples on multiplication of six), the construction of examples from a given number, the solution of examples with a blank, and others. To establish the connection between multiplication and division, and also to introduce the commutative property of multiplication, assignments requiring the construction of examples from three given numbers (for example, 6, 4, 24), and all other types of tasks mentioned above, are useful.

It is useful to conduct the check on the children's mastery of the tables in the form of an arithmetical dictation. Here, however, it is already possible to include the children themselves in the check, organizing classwork in pairs so pupils sitting next to each other check each other's work, and in case of doubt, check with the table or ask the teacher.

In studying multiplication and division by tables, it is very important to conduct numerous practice exercises requiring the solution of prepared examples. The children must in the end learn the tables by heart. For this reason, it is useful to drill them more than once in the reproduction of the tables' results.

To increase the number of examples solved, it is useful to make frequent use of the so-called half-written tasks, in which the children write down only the answers to the examples they solve, without rewriting the conditions in their notebooks. This form of work may be used when solving problems from the textbook, as well as work from individual cards and from variants written on the board.

Now let us go on to consider questions relating to instruction in solving problems when studying a given topic, The content and nature of the pupils' independent work on problems, in this case, are determined, to a significant degree, by the features of the problems under consideration. Here, as during the study of addition and subtraction, the primary goal of problem solving is the formation of important arithmetical concepts. In the process of forming these concepts, the differentiation of similar concepts and operations must be ensured.

This is also relevant to problems on division according to content, which acquaint the children with the application of familiar operations under new conditions--i.e., solving a practical problem which is different in principle from earlier ones. The solution of these problems causes a series of difficulties connected with precisely the necessity of distinguishing this application of division from division into different parts, which the children have been studying until this time. The distinction here is one of principle, but it also involves the form in which they are written.

The difficulties connected with the necessity of distinguishing similar concepts arise also in the consideration of problems on increasing and decreasing numbers by several times, and in the comparison of numbers through division.

The children often confuse increasing (decreasing) a number by several times with the familiar instance of increasing (decreasing) a number by several units; decreasing gets confused with increasing. The children sometimes multiply when they try to solve problems on comparison through division, just because in the question there is the word "bigger" ("How many times bigger?"); comparison by division also gets confused with comparison by subtraction.

All this requires the wide use of juxtaposition and opposition of similar concepts during independent exercises on the material of these problems. The juxtaposition and comparison of various types of problems can here be carried out in the most diverse concrete forms.

Here, as in the cases described above (relative to problems on addition and subtraction), the work sometimes aims at the clarification of the similarities, and sometimes especially at the clarification of the differences between the problems.

We will not cite here supplementary examples of this work--they may easily be composed by the teacher, analogous to those described above. We note only that they must lead to the further development of the knowledge, skills and habits which were formed by the material of earlier problems.

For example, while the conditions of problems requiring increasing (or decreasing) a number by several units were formerly written diagrammatically and the illustration was given through full use of visual aids with objects (the children had to draw the number of objects indicated by the conditions), now these forms are gradually replaced by a diagrammatic illustration in the form of strips or line segments, drawn at least approximately to scale.

Thus, illustration takes on a conditioned nature. While earlier it was directed toward helping the children develop a concrete, graphic idea of the conditions, this new type of graphic illustration reflects in a visual form the relationships among the quantities given in the problem. This is the next serious advance in the development of school children's visual concrete thought processes.

At first the teacher himself makes such drawings of the conditions of a problem analyzed in class, directs the children's attention to the method of their execution and requires them to reproduce the problem's conditions from this drawing. Later he increasingly includes in the children's independent work the formulation of problems from a drawing, and the construction of a drawing to represent the conditions of a given problem.

The formulation of a question for data, and the selection of data necessary to answer this question, are included in the assignments for independent work, as they were before. This work must also become gradually more complex. We cite a concrete example. The children are given the conditions and numerical data:

> On one day a store sold eight boxes of apples; on the second day it sold four.

The assignment is formulated thus:

> Formulate a question such that the problem is solved by addition; then change the question so that it is solved by division.

At this stage of instruction, it is necessary to assign the children increasingly more often, the task of independently constructing problems of a definite type. These assignments will be formulated as follows:

> Compose a problem on increasing a given number by several times; or compose a problem for whose solution it is necessary to use division according to content, etc.

In the opposite assignment, when it is necessary to select the proper numerical data for a given question, it is very important to use material from the children's own observations—everyday numerical data which they have encountered in solving the preceding problems from the textbook, numerical data drawn from class excursions, etc. If this material from life, which may be used as a basis for the construction of problems, is systematically accumulated, if these numbers are fixed, written in special notebooks, used for making posters, etc., all this material will help in organizing the children's independent work in class and will allow the teacher to vary this work, making the assignments simpler or more complex at his discretion.

Thus, the teacher can, for example, introduce a poster on which various postal rates are written; the children are asked to compose problems in which it is necessary to calculate how much more expensive a stamped envelope is than an unstamped one, or how much more expensive various types of telegrams are, etc. This assignment will be relatively easy for the children, since they can draw the necessary data directly from a consideration of the poster. Somewhat more complicated is this assignment:

> Using this poster, compose a problem on the comparison of numbers by division in which the precise numerical data to be used are not indicated.

This type of assignment becomes more complicated, if the teacher gives the children freedom to choose any subject, or any data from those in their notebooks.

The work described above involving the children's independent construction of problems will strengthen the link between arithmetic instruction and life. Aside from simple problems directed toward the formation of the concepts repeatedly mentioned above, the children's independent work must also include the solution of compound problems. These must be both problems of new types, and those which were solved before.

* * * *

Since we limited our consideration to the fundamental topics of the curricula for the first and second years of instruction, we naturally could not completely describe all the aspects of assignments for independent work, or all the methodological devices and forms of organization used in carrying out these tasks during arithmetic lessons.

We set ourselves the goal of merely giving examples to illustrate those topics which, during the course of the work, answer the requirements and goals, advanced in preceding chapters, for organizing children's independent work.

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